

# Contact non-squeezing at large scale in $\mathbb{R}^{2n} \times S^1$

Maia Fraser\*

July 8, 2016

## Abstract

We define a  $\mathbb{Z}_k$ -equivariant version of the cylindrical contact homology used by Eliashberg-Kim-Polterovich (2006) to prove contact non-squeezing for prequantized integer-capacity balls  $B(R) \times S^1 \subset \mathbb{R}^{2n} \times S^1$ ,  $R \in \mathbb{N}$  and we use it to extend their result to all  $R \geq 1$ . Specifically we prove if  $R \geq 1$  there is no  $\psi \in \text{Cont}(\mathbb{R}^{2n} \times S^1)$ , the group of compactly supported contactomorphisms of  $\mathbb{R}^{2n} \times S^1$  which squeezes  $\widehat{B}(R) = B(R) \times S^1$  into itself, i.e. maps the closure of  $\widehat{B}(R)$  into  $\widehat{B}(R)$ . A sheaf theoretic proof of non-existence of corresponding  $\psi \in \text{Cont}_0(\mathbb{R}^{2n} \times S^1)$ , the identity component of  $\text{Cont}(\mathbb{R}^{2n} \times S^1)$ , is due to Chiu (2014); it is not known if this is strictly weaker. Our construction has the advantage of retaining the contact homological viewpoint of Eliashberg-Kim-Polterovich and its potential for application in prequantizations of other Liouville manifolds. It makes use of the  $\mathbb{Z}_k$ -action generated by a vertical  $1/k$ -shift but can also be related, for prequantized balls, to the  $\mathbb{Z}_k$ -equivariant contact homology developed by Milin (2008) in her proof of orderability of lens spaces.

## 1 Introduction

Gromov's non-squeezing Theorem [12] identified a new rigidity phenomenon in symplectic geometry: the standard symplectic ball cannot be symplectically embedded (symplectically squeezed) into any cylinder of smaller radius.

By contrast, in the contact setting any (Darboux) ball can be contact embedded into an arbitrarily small neighbourhood of a point. Eliashberg-Kim-Polterovich [11] therefore studied, in the contact manifold  $\mathbb{R}^{2n} \times S^1$ , prequantized balls  $B^{2n} \times S^1$  and a more restrictive notion of squeezing: embedding via a globally defined compactly supported contactomorphism<sup>1</sup> which can further be required to be isotopic to the identity in this class. In their terminology, a (*contact*) *squeezing* of an open set  $U_1$  into an open set  $U_2$  is a compactly supported contact isotopy  $\{\phi_t\}_{t \in [0,1]}$  such that  $\phi_1(\text{Closure}(U_1)) \subset U_2$ . Constructed squeezings in [11] are of this kind. In their proofs of non-existence of a squeezing,

\*Department of Mathematics and Statistics, University of Ottawa

<sup>1</sup> In fact, a Darboux ball can be embedded into an arbitrarily small neighborhood of a point by a compactly supported contact isotopy, and  $B^{2n} \times S^1$ ,  $n > 1$  can be embedded into an arbitrarily small neighborhood of a point by a contactomorphism but in this case the embedding, while smoothly isotopic to the identity, is not compactly supported nor contact isotopic to the identity; see [11].

however, they prove a formally stronger statement, namely, non-existence of a compactly supported contactomorphism  $\psi$  (possibly not isotopic to the identity) such that  $\psi(\text{Closure}(U_1)) \subset U_2$ ; we will call such a contactomorphism a *coarse squeezing*. It is not known in  $\mathbb{R}^{2n} \times S^1$  if this formally stronger non-squeezing statement is strictly stronger.

Eliashberg-Kim-Polterovich [11] showed contact squeezing is closely related to the concept of orderability<sup>2</sup> of a contact manifold introduced by Eliashberg-Polterovich in [10] and moreover, squeezing may be possible at one scale and not at another in the same manifold. More precisely, consider  $\mathbb{R}^{2n} \times S^1$  with contact structure  $\ker(dt - \alpha_L)$  and Liouville form  $\alpha_L = \frac{1}{2}(ydx - xdy)$  on  $\mathbb{R}^{2n}$ ,  $x, y \in \mathbb{R}^{2n}$ ,  $t \in S^1$ . Let  $\text{Cont}_0(\mathbb{R}^{2n} \times S^1)$  denote the identity component of the group  $\text{Cont}(\mathbb{R}^{2n} \times S^1)$  of compactly supported contactomorphisms. These are time-1 maps of compactly supported contact isotopies, so squeezing of  $U_1$  into  $U_2$  is equivalent to the existence of  $\phi \in \text{Cont}_0(\mathbb{R}^{2n} \times S^1)$  such that  $\phi(\text{Closure}(U_1)) \subset U_2$ . For any positive  $R$ , let  $B(R) := \{w \in \mathbb{R}^{2n} : \pi|w|^2 < R\}$  be the ball of symplectic capacity  $R$ , and  $\widehat{B}(R) := B(R) \times S^1$  its prequantization. Eliashberg-Kim-Polterovich proved:

**Theorem 1.1** (Eliashberg-Kim-Polterovich [11]). *Let  $R < 1$ . Then there is a contact squeezing of  $\widehat{B}(R)$  into itself. By contrast, if  $R \geq 1$  is an integer, there is no coarse contact squeezing of  $\widehat{B}(R)$  into itself.*

It remained unknown for some time whether non-squeezing in this setting held also for non-integer  $R > 1$ . In 2010 Tamarkin [18] sketched a proof of the affirmative answer. This was recently proved by Chiu:

**Theorem 1.2** (Chiu [7]). *Let  $R \geq 1$ . Then there is no contact squeezing of  $\widehat{B}(R)$  into itself.*

Discussions with Tamarkin [18] also inspired our use of  $\mathbb{Z}_k$ -equivariance: we prove Theorem 1.2 by means of a  $\mathbb{Z}_k$ -equivariant version of the cylindrical contact homology of Eliashberg-Kim-Polterovich [11]. In fact, as in [11], we prove a formally stronger statement:

**Theorem 1.2’.** *Let  $R \geq 1$ . Then there is no coarse contact squeezing of  $\widehat{B}(R)$  into itself.*

In Section 4, we observe that Theorem 1.2 also has consequences for squeezings of  $\widehat{B}(R)$  into itself when  $R < 1$ : in some cases they require a larger domain of support, i.e., *squeezing room*, than previously established. This is stated in Theorem 4.1.

**Remark 1.3.** *Squeezing of  $\widehat{B}(R)$  into itself is an open condition. Indeed, existence of a squeezing of  $\widehat{B}(R)$  into itself implies existence of a squeezing of some*

---

<sup>2</sup>Given a Liouville domain  $(M, \omega, L)$  with ideal contact boundary  $P$  (see Section 1.5 of [11] for terminology), non-squeezing at arbitrarily small scales for “fiberwise star-shaped domains” in  $M \times S^1$  implies orderability of  $P$ .

larger  $\widehat{B}(R_1)$  into a smaller  $\widehat{B}(R_2)$  and this will in particular squeeze all intermediate prequantized balls into themselves. An equivalent<sup>3</sup> formulation of Theorem 1.2 (resp. 1.2') is therefore : let  $1 < R_2 < R_1$ , then there is no squeezing (resp. coarse squeezing) of  $\widehat{B}(R_1)$  into  $\widehat{B}(R_2)$ . Note the strict inequality  $1 < R_2$  which can without loss of generality be imposed.

## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Squeezing vs. <math>\mathbb{Z}_k</math>-equivariant squeezing</b>	<b>3</b>
<b>3</b>	<b><math>\mathbb{Z}_k</math>-equivariant contact homology <math>CH_*^{\mathbb{Z}_k}(-)</math></b>	<b>5</b>
3.1	Results for $\widehat{B}(R)$ : application to non-squeezing . . . . .	13
3.2	Computations for $\widehat{B}(R)$ . . . . .	14
<b>4</b>	<b>Squeezing room and non-squeezing</b>	<b>21</b>

## 2 Squeezing vs. $\mathbb{Z}_k$ -equivariant squeezing

We prove Theorem 1.2' by proving an alternate, equivalent statement, Theorem 2.1. To formulate this, let  $\text{Cont}_0^{\mathbb{Z}_k}(\mathbb{R}^{2n} \times S^1)$  denote the identity component of the group  $\text{Cont}^{\mathbb{Z}_k}(\mathbb{R}^{2n} \times S^1)$  of compactly supported  $\mathbb{Z}_k$ -equivariant contactomorphisms for the  $\mathbb{Z}_k$ -action generated by a vertical shift  $\nu : (x, y, t) \mapsto (x, y, t + 1/k)$ . By analogy with the definition of contact squeezing [11], define a  $\mathbb{Z}_k$ -equivariant (contact) squeezing of an open set  $U_1$  into an open set  $U_2$  as  $\phi \in \text{Cont}_0^{\mathbb{Z}_k}(\mathbb{R}^{2n} \times S^1)$  such that  $\phi(\text{Closure}(U_1)) \subseteq U_2$ . Define a coarse  $\mathbb{Z}_k$ -equivariant squeezing analogously but requiring only  $\phi \in \text{Cont}^{\mathbb{Z}_k}(\mathbb{R}^{2n} \times S^1)$ . Our main result is:

**Theorem 2.1.** *For any prime  $k \in \mathbb{N}$  and any  $\ell \in \mathbb{N}$  such that  $\ell < k$ , if  $R_2 < 1/\ell < R_1$ , there is no  $\mathbb{Z}_k$ -equivariant contact squeezing of  $\widehat{B}(R_1)$  into  $\widehat{B}(R_2)$ , not even a coarse one.*

**Remark 2.2.** *Note that case  $k = 2$  of Theorem 2.1 is implied by the non-existence of a squeezing of  $\widehat{B}(1)$  into itself (proved by [11], see Theorem 1.1 above). It will therefore be sufficient for the purposes of proving Theorem 2.1 to establish the case  $k > 2$  and we make this assumption in the computations of Section 3.2.*

The equivalence of Theorems 1.2' and 2.1 follows from properties of the contact  $k$ -fold cover of  $\mathbb{R}^{2n} \times S^1$ . Indeed, let  $S^1 = \mathbb{R}/\mathbb{Z}$  and, as a manifold, define the  $k$ -fold cover of  $\mathbb{R}^{2n} \times S^1$  by the covering map

$$\begin{aligned} \tau : \mathbb{R}^{2n} \times S^1 &\rightarrow \mathbb{R}^{2n} \times S^1 \\ (z, t) &\mapsto (\sqrt{k}z, kt). \end{aligned}$$

---

<sup>3</sup> Theorems 1.2 and 1.2' refer to existence of a squeezing only; the required squeezing room could in principle be greater the farther apart  $R_1$  and  $R_2$  are taken.

Assuming the standard contact structure on both base and cover, one has that  $\tau$  is a contactomorphism. Deck transformations in the cover are then also contactomorphisms; they form a cyclic group isomorphic to  $\mathbb{Z}_k$ , generated by  $\nu$ . We have:

**Lemma 2.3.**  $\text{Cont}^{\mathbb{Z}_k}(\mathbb{R}^{2n} \times S^1) = \{\tilde{\phi} : \phi \in \text{Cont}(\mathbb{R}^{2n} \times S^1)\}$  where  $\tilde{\phi}$  is the unique lift (as a compactly supported diffeomorphism) of  $\phi$ . Likewise we have  $\text{Cont}_0^{\mathbb{Z}_k}(\mathbb{R}^{2n} \times S^1) = \{\tilde{\phi} : \phi \in \text{Cont}_0(\mathbb{R}^{2n} \times S^1)\}$ . Moreover, for any  $R_1, R_2 > 0$ ,  $\tilde{\phi}(\widehat{B}(R_1/k)) \subseteq \widehat{B}(R_2/k) \Leftrightarrow \phi(\widehat{B}(R_1)) \subseteq \widehat{B}(R_2)$ .

*Proof.* Each  $\phi \in \text{Cont}(\mathbb{R}^{2n} \times S^1)$  has a unique lift<sup>4</sup> to a compactly supported diffeomorphism  $\tilde{\phi}$  of the  $k$ -fold cover and this  $\tilde{\phi}$  is a contactomorphism since  $\phi$  is and the contact structure in the cover is the pullback by  $\tau$  of its counterpart in the base. Moreover,  $\tilde{\phi}$  is  $\mathbb{Z}_k$ -equivariant by construction since  $\nu$  is a deck transformation. Thus  $\tilde{\phi} \in \text{Cont}^{\mathbb{Z}_k}(\mathbb{R}^{2n} \times S^1)$ . On the other hand any element  $\phi' \in \text{Cont}^{\mathbb{Z}_k}(\mathbb{R}^{2n} \times S^1)$  descends to a well-defined contactomorphism  $\phi$  of the base since it commutes with  $\nu$ , and so we have  $\phi' = \tilde{\phi}$ . Applying this correspondence to isotopies we obtain the statement regarding  $\text{Cont}_0^{\mathbb{Z}_k}(\mathbb{R}^{2n} \times S^1)$  and  $\text{Cont}_0(\mathbb{R}^{2n} \times S^1)$ . The final statement of the Lemma is immediate because  $\tilde{\phi}(\tau^{-1}(\mathcal{U})) \subseteq \tau^{-1}(\mathcal{V}) \Leftrightarrow \phi(\mathcal{U}) \subseteq \mathcal{V}$ .  $\square$

As a Corollary, we obtain the claimed equivalence of Theorems 1.2' and 2.1:

**Corollary 2.4.**  $\widehat{B}(R)$  can be squeezed into itself by  $\phi \in \text{Cont}(\mathbb{R}^{2n} \times S^1)$  if and only if  $\widehat{B}(R/k)$  can be squeezed into itself by  $\tilde{\phi} \in \text{Cont}^{\mathbb{Z}_k}(\mathbb{R}^{2n} \times S^1)$  (and likewise for  $\text{Cont}_0(\mathbb{R}^{2n} \times S^1)$  and  $\text{Cont}_0^{\mathbb{Z}_k}(\mathbb{R}^{2n} \times S^1)$ ). Thus, Theorem 1.2' is equivalent to Theorem 2.1.

*Proof.* The first statement is immediate from Lemma 2.3. For the equivalence of Theorems note that, once  $k$  prime and  $\ell < k$  are fixed,  $R_2$  in the hypotheses of Theorem 2.1 can without loss of generality be specified to be arbitrarily close to  $1/\ell$ , in particular such that  $1/k < R_2 < 1/\ell < R_1$  since squeezing  $\widehat{B}(R_1)$  into a smaller  $\widehat{B}(R_2)$  would imply squeezing  $\widehat{B}(R_1)$  into all  $\widehat{B}(R)$  for  $R_2 \leq R < R_1$ . Therefore, Theorem 2.1 is equivalent (putting  $R'_i = kR_i$ ) to non-existence - under the hypothesis  $k$  prime,  $\ell < k$ , and  $1 < R'_2 < k/\ell < R'_1$  - of  $\phi \in \text{Cont}(\mathbb{R}^{2n} \times S^1)$  which squeezes  $\widehat{B}(R'_1)$  into  $\widehat{B}(R'_2)$ . Since for every pair  $R'_2 < R'_1$  there exist<sup>5</sup>  $k, \ell \in \mathbb{N}$  with  $k$  prime such that  $R'_2 < k/\ell < R'_1$  the above can be restated as non-existence of a squeezing of  $\widehat{B}(R'_1)$  into  $\widehat{B}(R'_2)$  when  $1 < R'_2 < R'_1$ . By Remark 1.3 this is equivalent to Theorem 1.2'.  $\square$

<sup>4</sup>For shorthand, write  $\tau : C \rightarrow B$  where both cover  $C$  and base  $B$  are  $\mathbb{R}^{2n} \times S^1$  and  $\phi : B \rightarrow B$  is a compactly supported diffeomorphism. Lift  $\phi$  to the local diffeomorphism  $\hat{\phi} : C \rightarrow B$  given by  $\hat{\phi} := \phi \circ \tau$ . Because  $B = \mathbb{R}^{2n} \times S^1$  and  $\phi$  has compact support,  $\hat{\phi}$  acts trivially on  $\pi_1(B)$  so  $\hat{\phi}$  maps  $\pi_1(C)$  to the same subgroup of  $\pi_1(B)$  as does  $\tau$ , namely  $k\mathbb{Z} \subset \mathbb{Z} = \pi_1(B)$  and, hence, by the unique lifting property, lifts to  $\tilde{\phi} : C \rightarrow C$  such that  $\tau\tilde{\phi}(z) = \hat{\phi}(z) = \phi\tau(z)$  which is unique up to composition with deck transformations; of such lifts there is a unique one with compact support.

<sup>5</sup>Put  $\beta = R_1/R_2$  and consider intervals  $I_\ell = (\ell R_2, \ell \beta R_2)$ ,  $\ell \in \mathbb{N}$ . Since  $\beta > 1$ , for sufficiently large  $\ell$  each  $I_\ell$  overlaps with  $I_{\ell+1}$  and so by the infinitude of primes  $\exists \ell \in \mathbb{N}$  such that  $I_\ell$  contains a prime  $k$ ; i.e.,  $R_2 < k/\ell < R_1$ .

**Remark 2.5** (The implicit role of  $\mathbb{Z}_k$ -equivariance in [11]). *Eliashberg-Kim-Polterovich [11] use (non-equivariant) cylindrical contact homology to prove that  $\widehat{B}(1)$  cannot be squeezed into itself. They then conclude by a covering space argument that  $\widehat{B}(m)$  cannot be squeezed into itself for any  $m \in \mathbb{N}$ . Their two-part proof does not emphasize the inherent  $\mathbb{Z}_m$ -action; however, in the language of the present section, it amounts to first showing  $\widehat{B}(1)$  cannot be squeezed into itself, noting this implies  $\widehat{B}(1)$  cannot be squeezed into itself  $\mathbb{Z}_m$ -equivariantly, and then using the last statement of Lemma 2.3 to conclude  $\widehat{B}(m)$  cannot be squeezed into itself. For the prequantized ball  $\widehat{B}(1)$ , both  $\mathbb{Z}_k$ -equivariant squeezing and (non-equivariant) squeezing are impossible and, moreover,  $\widehat{B}(1)$  is the only prequantized ball for which the contact homology of [11] can directly<sup>6</sup> rule out squeezing. A priori, however, non-existence of a  $\mathbb{Z}_m$ -equivariant squeezing is a weaker notion than non-existence of a squeezing and could potentially be true in more situations. Indeed, this is the contribution of the present paper. For  $k$  prime, we use  $\mathbb{Z}_k$ -equivariant contact homology to rule out  $\mathbb{Z}_k$ -equivariant squeezing of any  $\widehat{B}(1/\ell)$ ,  $\ell \in \mathbb{N}$  into itself when  $\ell < k$  (Theorem 2.1), although such prequantized balls are known by [11] to be squeezable into themselves non-equivariantly (c.f. Theorem 1.1).*

### 3 $\mathbb{Z}_k$ -equivariant contact homology $CH_*^{\mathbb{Z}_k}(-)$

We now define  $CH_*^{\mathbb{Z}_k}(-)$ , a  $\mathbb{Z}_k$ -equivariant analog of the (non-equivariant) cylindrical contact homology  $CH_*(-)$  developed by Eliashberg-Kim-Polterovich [11] and Kim [15] for  $\mathbb{R}^{2n} \times S^1$ , which also has similarities to a  $\mathbb{Z}_k$ -equivariant version of  $CH_*(-)$  developed by Milin [16] for a different  $\mathbb{Z}_k$ -action. Like these theories, it lies within the general framework of symplectic field theory proposed by Eliashberg-Givental-Hofer [9].

We give the construction of  $CH_*^{\mathbb{Z}_k}(-)$  as we recall that of  $CH_*(-)$ , and we explain how well-definedness of  $CH_*^{\mathbb{Z}_k}(-)$  follows with only minor modifications from corresponding arguments for  $CH_*(-)$  due to [4], [8], [3], [13] with, in addition, the construction of coherent orientations from Bourgeois-Mohnke [5]; these aspects are highlighted in boldface in the construction and addressed in the paragraph “Technical arguments”.

Foundational issues with cylindrical contact homology which arise in the presence of multiply covered orbits (see for example [14] for a discussion) are avoided in both our setting and that of Eliashberg-Kim-Polterovich [11] by taking as generators only closed Reeb orbits in the free homotopy class  $[\text{pt} \times S^1]$ .

In Section 3.1 we state our main results for  $CH_*^{\mathbb{Z}_k}(\widehat{B}(R))$ , showing how they imply non-squeezing. In Section 3.2 we give proofs of these statements as well as a proof of analogous statements for  $CH_*(\widehat{B}(R))$  with  $\mathbb{Z}_k$ -coefficients. Our aim is for the present paper to serve as an extension of Eliashberg-Kim-Polterovich [11] in which the reader familiar with [11] can view non-squeezing at large scale from the vantage point provided by contact homology.

---

<sup>6</sup>The  $\mathbb{Z}$ -graded vector spaces  $CH_*(\widehat{B}(R))$  are isomorphic for all  $R > 1$  and differ from  $CH_*(\widehat{B}(r))$  for  $r < 1$ . This makes the invariants suitable to directly rule out only squeezing of  $\widehat{B}(R)$  into  $\widehat{B}(r)$  (since we allow squeezings to have arbitrarily large support - see Remark 3.7).

**Remark 3.1.** *To keep the presentation compact: we assume  $k > 2$  from now on. Our construction applies equally well to the case  $k = 2$  but this would require specification of a different projective resolution to compute equivariant homology (see Remark 3.2) and result in slightly different<sup>7</sup> chain complexes in equivariant computations. Since we do not need the case  $k = 2$  (see Remark 2.2) we omit it.*

We restrict attention to  $V := \mathbb{R}^{2n} \times S^1$  with contact structure  $\ker(dt - \alpha_L)$  and  $\mathbb{Z}_k$ -action as defined in Sections 1 and 2. Our construction, however, like that of [11], goes through verbatim in prequantizations  $M \times S^1$  for many other Liouville manifolds  $M$  (c.f. Theorem 4.47 of [11]).

An open domain  $U \subset V$  with compact closure is said to be *fiberwise star-shaped* [11] if its boundary  $\partial U$  is transverse to the fibers  $M \times \{t\}$ ,  $t \in S^1$  and intersects them along hypersurfaces transverse to the Liouville vector field  $L$  determined by  $\alpha_L = i_L \omega$ . In particular prequantized balls  $\widehat{B}(R)$  are fiberwise star-shaped. Let  $\mathcal{U}$ , resp.  $\mathcal{U}_k$ , be the class of domains  $\psi(U)$  such that  $U$  is fiberwise star-shaped, resp. fiberwise star-shaped and  $\mathbb{Z}_k$ -invariant, and  $\psi \in \text{Cont}(\mathbb{R}^{2n} \times S^1)$ , resp.  $\psi \in \text{Cont}^{\mathbb{Z}_k}(\mathbb{R}^{2n} \times S^1)$ . Given  $U \in \mathcal{U}_k$ , we construct the  $\mathbb{Z}$ -graded vector space  $CH_*^{\mathbb{Z}_k}(U)$ , as [11] constructed  $CH_*(U)$  for  $U \in \mathcal{U}$ , so that the resulting association is functorial in an invariant way (invariant under the action of  $\text{Cont}^{\mathbb{Z}_k}(\mathbb{R}^{2n} \times S^1)$  resp.  $\text{Cont}(\mathbb{R}^{2n} \times S^1)$  - see Theorem 3.3 below, resp. Theorem 4.47 of [11]).

**Admissible forms.** Denote by  $\mathcal{F}_{ad}(U)$  the set of all *admissible* contact forms on  $U$ , namely forms  $\lambda = F(dt - \alpha_L)$  where  $F$  is a positive Hamiltonian equal to a constant  $K$  outside a compact set, and the Reeb flow of  $\lambda$  has no contractible closed orbits  $\gamma$  of action (i.e. period)  $\mathcal{A}(\gamma) := \int_\gamma \lambda$  less than or equal to  $K$ . Let  $\mathcal{F}_{ad}(U, \epsilon) \subset \mathcal{F}_{ad}(U)$  consist of those forms which do not have  $\epsilon$  as *critical value*, i.e. have no closed Reeb orbit  $\gamma$  of action  $\mathcal{A}(\gamma) = \epsilon$ . Denote by  $\mathcal{F}_{ad}^{\mathbb{Z}_k}(U, \epsilon)$  those  $\lambda \in \mathcal{F}_{ad}(U, \epsilon)$  which are  $\mathbb{Z}_k$ -invariant. As in [11], endow these spaces of contact forms with the “anti-natural” partial order  $\preceq$ :

$$\lambda'' \preceq \lambda' \Leftrightarrow \lambda'' \geq \lambda'.$$

An admissible contact form  $\lambda$  equal to  $K(dt - \alpha_L)$  outside a compact set is said to be *regular* if all orbits  $\gamma$  in

$$\mathcal{P}_\lambda := \{\gamma : \mathcal{A}(\gamma) < K, [\gamma] = [\text{pt} \times S^1]\}$$

are *non-degenerate*. Here  $[\gamma]$  denotes the free homotopy class of  $\gamma$  as a loop  $S^1 \rightarrow V$  and non-degeneracy means the restriction of the Poincaré return map for the Reeb flow of  $\lambda$  along  $\gamma$  to the contact hyperplane bundle does not have 1 as eigenvalue. By standard arguments generic admissible contact forms are regular.

For regular  $\lambda$ ,  $\mathcal{P}_\lambda$  is graded by the Conley-Zehnder index  $\mu_{CZ}(\gamma)$  defined in terms of paths of symplectic matrices as in [17]. Note this convention differs by  $-n$  from that in [11] (c.f. Erratum to [11]). If  $\lambda$  is  $\mathbb{Z}_k$ -invariant, then  $\gamma \mapsto \nu\gamma$  generates a  $\mathbb{Z}_k$ -action on  $\mathcal{P}_\lambda$  which preserves  $\mu_{CZ}$ . Using  $\lambda$ , identify the

<sup>7</sup>In all these complexes multiplication by  $(T - 1)$  must be replaced with multiplication by  $(T + 1)$ .

symplectization  $W$  of  $V$  with  $(V \times \mathbb{R}, d(e^s \lambda))$  and put  $W_* := (V \setminus \{0\} \times S^1) \times \mathbb{R} \subset W$ . Let  $\xi$  and  $\tau$  denote the hyperplane bundles on  $W_*$  resp.  $W$  which are respectively pull-backs of the standard contact structure on  $S^{2n-1}$  and  $\ker(\lambda)$  on  $V$ . Lift also the  $\mathbb{Z}_k$ -action of  $V$  to  $W$ ,  $W_*$ . Write  $R_\lambda$  for the Reeb vector field of  $\lambda$ . Consider almost complex structures  $J$  on  $V \times \mathbb{R}$  which are *adjusted* to  $\lambda$  in the sense of [11]:  $J$  is invariant under translations in the  $\mathbb{R}$ -coordinate,  $J\tau = \tau$  and  $J|_\tau$  is compatible with  $d\lambda$ ,  $J(\frac{\partial}{\partial s}) = R_\lambda$ , and finally,  $J\xi = \xi$  outside the symplectization of a compact subset of  $V$ . When  $\lambda$  is  $\mathbb{Z}_k$ -invariant, a  $J$  adjusted to  $\lambda$  which is also  $\mathbb{Z}_k$ -invariant is said to be  $\mathbb{Z}_k$ -adjusted to  $\lambda$ .

**Chain complex.** Given an almost complex structure  $J$  adjusted, resp.  $\mathbb{Z}_k$ -adjusted, to a regular admissible  $\lambda$ , let  $C(\lambda, J)$ , resp.  $C^{\mathbb{Z}_k}(\lambda, J)$ , be the vector space generated over  $\mathbb{Z}_k$  by orbits in  $\mathcal{P}_\lambda$  and  $\mathbb{Z}$ -graded by  $\mu_{CZ}$ . For  $a, b$  which are not critical values of  $\lambda$ , using the action filtration let

$$C^{(a,b)}(\lambda, J) = C(\lambda, J) \cap \text{span}\{\gamma : \mathcal{A}(\gamma) < b\} / \text{span}\{\gamma : \mathcal{A}(\gamma) > a\}$$

and define  $C^{\mathbb{Z}_k, (a,b)}(\lambda, J)$  analogously. To define a differential  $d$  we make no changes to the definition of [11], however **transversality** arguments needed to establish well-definedness of  $d$  for generic  $J$  require a slight modification as explained below. Note: the definition of the differential in [11] is given in more generality, as part of the construction of generalized Floer homology; for a more accessible,  $CH$ -specific description the reader is referred to [2]. Roughly speaking, for both equivariant and non-equivariant theories we consider for closed Reeb orbits  $\gamma_\pm$  the moduli space  $\hat{\mathcal{M}}(\gamma_+, \gamma_-)$  consisting of  $J$ -holomorphic cylinders

$$G = (g, a) : (\mathbb{R} \times S^1, j) \rightarrow (W = V \times \mathbb{R}, J)$$

asymptotic at the ends to  $\gamma_\pm$  such that  $\lim_{s \rightarrow \pm\infty} f(s, t) = \gamma_\pm(T_\pm t)$  and  $\lim_{s \rightarrow \pm\infty} a(s, t) = \pm\infty$ , for  $T_\pm$  the periods of  $\gamma_\pm$  respectively, and assuming standard complex structure  $j$  on  $\mathbb{R} \times S^1$  and almost complex structure  $J$  adjusted to  $\lambda$  on  $W$ . In the non-equivariant case, for generic almost complex structures adjusted to  $\lambda$  the moduli space  $\hat{\mathcal{M}}(\gamma_+, \gamma_-)$  forms a smooth oriented manifold. In the equivariant case, we allow only  $\mathbb{Z}_k$ -adjusted complex structures and modify the usual genericity argument as explained below.

This manifold  $\hat{\mathcal{M}}(\gamma_+, \gamma_-)$ , in both equivariant and non-equivariant cases, is acted on by the group  $\mathbb{R} \times (\mathbb{R} \times S^1)$  of holomorphic re-parametrizations (of target, domain) and quotienting by these one obtains a smooth oriented manifold  $\mathcal{M}(\gamma_+, \gamma_-)$  of dimension  $\mu_{CZ}(\gamma_+) - \mu_{CZ}(\gamma_-) - 1$ . Since  $J$  and  $\lambda$  are standard at infinity and  $\lambda$  has no contractible closed Reeb orbits with action  $\leq K$ , **compactness** results of [4] show that  $\mathcal{M}(\gamma_+, \gamma_-)$  compactifies to a moduli space of “broken”  $J$ -holomorphic cylinders; this applies in both the equivariant and non-equivariant frameworks. In the case  $\mu_{CZ}(\gamma_+) - \mu_{CZ}(\gamma_-) = 1$  this moduli space is a compact, oriented 0-dimensional manifold and so consists of a finite number of points with sign. For this, one needs to have arranged a system of **coherent orientations** on moduli spaces in a way which is compatible with glueing, and, in the case of our  $\mathbb{Z}_k$ -equivariant theory, invariant under the action of  $\mathbb{Z}_k$ . This extra property comes for free from the construction of [5] assuming the contact forms and almost complex structures used are  $\mathbb{Z}_k$ -invariant (see below). The differential on  $C(\lambda, J)$ , resp.  $C^{\mathbb{Z}_k}(\lambda, J)$  is then defined by counting, modulo  $k$ , all points

in  $\mathcal{M}(\gamma_+, \gamma_-)$  with signs. By construction of the compactification [13] it follows that  $d^2 = 0$  so  $(C(\lambda, J), d)$ , resp.  $(C^{\mathbb{Z}_k}(\lambda, J), d)$ , is a chain complex. Moreover, since  $G \in \mathcal{M}(\gamma_+, \gamma_-)$  if and only if  $\nu G \in \mathcal{M}(\nu\gamma_+, \nu\gamma_-)$  and signs of elements of  $\mathcal{M}(\gamma_+, \gamma_-)$  are preserved by  $\nu$ , it follows that  $(C^{\mathbb{Z}_k}(\lambda, J), d)$  comes equipped with a  $\mathbb{Z}_k$ -action, i.e.,  $d$  is  $\mathbb{Z}_k$ -equivariant for the  $\mathbb{Z}_k$ -action induced on spaces of Reeb orbits by  $\nu$ .  $CH_*^{(a,b)}(\lambda)$  is defined as the homology of  $(C^{(a,b)}(\lambda, J), d)$  after showing it does not depend on  $J$ . Likewise we define  $CH_*^{\mathbb{Z}_k, (a,b)}(\lambda)$  as the  $\mathbb{Z}_k$ -equivariant homology of  $(C^{\mathbb{Z}_k, (a,b)}(\lambda, J), d)$ . Note that any contactomorphism, resp.  $\mathbb{Z}_k$ -equivariant contactomorphism,  $\psi$  will set up a 1-1 correspondence not only between chain groups for  $\lambda$  and  $\psi_*\lambda$  but also between moduli spaces defined above so there is a chain map, resp.  $\mathbb{Z}_k$ -equivariant chain map,  $\psi_\#$  from the respective chain complex for  $\lambda$  to that for  $\psi_*\lambda$  which is an isomorphism in all degrees, yielding (grading-preserving) isomorphisms:  $\psi_\# : CH_*^{(a,b)}(\lambda) \xrightarrow{\cong} CH_*^{(a,b)}(\psi_*\lambda)$  and  $\psi_\# : CH_*^{\mathbb{Z}_k, (a,b)}(\lambda) \xrightarrow{\cong} CH_*^{\mathbb{Z}_k, (a,b)}(\psi_*\lambda)$ .

**Remark 3.2. (Equivariant homology computation)** *In general, given a  $\mathbb{Z}_k$ -action on a chain complex  $(C_*, d)$ ,  $\mathbb{Z}_k$ -equivariant homology is defined as follows. Let  $\mathcal{R} = \mathbb{Z}_k[\mathbb{Z}_k] \cong \mathbb{Z}_k[T]/(T^k - 1)$  be the group ring of the group  $\mathbb{Z}_k$  with  $\mathbb{Z}_k$ -coefficients. Let  $\mathbb{Z}_k$  act on  $\mathbb{Z}_k$  trivially to make  $\mathbb{Z}_k$  into an  $\mathcal{R}$ -module. Let  $(E_*, \delta)$  be any projective resolution of  $\mathbb{Z}_k$  (as an  $\mathcal{R}$ -module) and tensor  $(E_*, \delta)$  with  $(C_*, d)$  over  $\mathcal{R}$ . The  $\mathbb{Z}_k$ -equivariant homology of  $(C_*, d)$  is defined to be the usual homology of this tensor product. Up to isomorphism this is independent of the choice of  $(E_*, \delta)$  since all projective resolutions are quasi-isomorphic as  $\mathcal{R}$ -chain complexes (to  $0 \rightarrow \mathcal{R} \rightarrow 0$  and so to each other) and thus the resulting tensor products are also quasi-isomorphic. In computations, we follow Milin [16] and - assuming  $k > 2$  - use the projective resolution*

$$\dots \mathcal{R} \xrightarrow{\cdot(T-1)} \mathcal{R} \xrightarrow{\cdot(T^{k-1} + \dots + 1)} \mathcal{R} \xrightarrow{\cdot(T-1)} \mathcal{R} \xrightarrow{\cdot(T^{k-1} + \dots + 1)} \mathcal{R} \xrightarrow{\cdot(T-1)} \mathcal{R} \rightarrow 0.$$

**Monotonicity morphisms.** Given  $\lambda_- < \lambda_+$ , [11] define<sup>8</sup> a monotonicity chain map  $\text{mon} : C^{(a,b)}(\lambda_+, J) \rightarrow C^{(a,b)}(\lambda_-, J)$  by considering the moduli space  $\hat{\mathcal{M}}(\gamma_+, \gamma_-)$  of  $J$ -holomorphic cylinders between  $\gamma_\pm$  which are closed Reeb orbits for  $\lambda_\pm$  respectively, where  $J$  is an almost complex structure “adjusted to an admissible concordance structure” between  $\lambda_\pm$ . This construction is then extended to the case  $\lambda_- \leq \lambda_+$ , i.e.,  $\lambda_+ \preceq \lambda_-$ . In our equivariant setting we follow the identical procedure using however  $\mathbb{Z}_k$ -invariant ingredients.

To describe this more precisely, assume  $\lambda_- < \lambda_+$  where  $\lambda_\pm = F_\pm(dt - \alpha_L)$ ,  $F_\pm > 0$ . Let  $a_- < a_+ \in \mathbb{R}$  and put  $W_{a_-}^{a_+} = V \times [a_-, a_+]$  with coordinates  $(v, s)$ . Consider any function  $F : W_{a_-}^{a_+} \rightarrow \mathbb{R}_+$  such that  $\frac{\partial F}{\partial s} > 0$ , and, outside a compact subset of  $V$ ,  $F$  depends only on  $s$ , while near the respective boundaries,  $s = a_\pm$ ,  $F$  is linear in  $s$  of the form  $F(v, s) = (1 + s - a_\pm)F_\pm(v)$ . In particular,  $F(v, a_\pm) = F_\pm(v)$ . Using such an  $F$ ,  $W_{a_-}^{a_+}$  can be identified with the region  $\{F_-(v) \leq r \leq F_+(v)\}$ ,  $(v, r) \in V \times \mathbb{R}_+ = W$  via the map  $\Phi_F : (v, s) \mapsto (v, F(v, s))$ . View  $V \times \mathbb{R}_+$  as the symplectization of  $V$  with symplectic form  $d(r(dt - \alpha_L))$  and denote the pull-back of this 2-form to  $\text{Int}(W_{a_-}^{a_+})$  as  $\omega_F := d(F(dt - \alpha_L))$ . The pair  $(W_{a_-}^{a_+}, F(dt - \alpha_L))$  is said to be an (admissible) concordance structure between

<sup>8</sup>(translated to contact structures from the language of Hamiltonian structures in [11])



$\lambda_{\pm}$ , or in the equivariant setting an (*admissible*)  $\mathbb{Z}_k$ -concordance structure if  $F$  is also  $\mathbb{Z}_k$ -invariant (for the lifted  $\mathbb{Z}_k$ -action). The linearity of  $F$  in  $s$  near the boundaries,  $s = a_{\pm}$ , of  $W_{a_{\pm}}^{a+}$  (c.f. “normal form” in [11]) gives neighbourhoods of these boundaries already locally the structure of symplectizations of  $(V, \lambda_{\pm})$  respectively. By extending  $F$  linearly to all of  $\mathbb{R}$  and extending  $\omega_F$  accordingly,  $(\overline{W} = V \times \mathbb{R}, \omega_F)$  acquires the structure of a symplectic cobordism between  $(V, \lambda_-)$  and  $(V, \lambda_+)$ . Moreover, when  $F$  is  $\mathbb{Z}_k$ -invariant so is the symplectic structure  $\omega_F$  on  $\overline{W}$ . Warning: though  $W = \overline{W} = V \times \mathbb{R}$ , we use different names to recall the different symplectic structures -  $(\overline{W}, \omega_F)$  is a cobordism between  $(V, \lambda_{\pm})$  while  $(W, d(e^s \lambda))$  is the symplectization of  $(V, \lambda)$ .

An almost complex structure  $J$  on  $W_{a_{\pm}}^{a+}$  is *adjusted* to the concordance  $(W_{a_{\pm}}^{a+}, F(dt - \alpha_L))$  between  $\lambda_{\pm}$  for specific choices,  $J_{\pm}$ , of respectively adjusted almost complex structures on symplectizations  $(W, d(e^s \lambda_{\pm}))$  of  $(V, \lambda_{\pm})$  if  $\omega_F$  tames  $J$ ,  $J$  is pseudoconvex at infinity and  $J$  agrees with  $J_{\pm}$  near the boundaries  $s = a_{\pm}$ . The second condition means that  $\overline{W}$  is foliated by weakly  $J$ -convex hypersurfaces outside the symplectization of a compact subset of  $V$ . In the equivariant setting, an adjusted  $J$  which is also  $\mathbb{Z}_k$ -invariant is said to be  $\mathbb{Z}_k$ -adjusted. In either case, we denote also by  $J$  the extension of  $J$  to  $\overline{W}$ . Analogous to when we defined the differential  $d$ , given  $\lambda_- < \lambda_+$  we consider in both equivariant and non-equivariant theories the moduli space  $\hat{\mathcal{M}}(\gamma_+, \gamma_-)$  consisting of  $J$ -holomorphic cylinders

$$G = (f, a) : (\mathbb{R} \times S^1, j) \rightarrow (\overline{W} = V \times \mathbb{R}, J)$$

asymptotic at the ends to  $\gamma_{\pm}$  such that  $\lim_{s \rightarrow \pm\infty} f(s, t) = \gamma_{\pm}(T_{\pm}t)$  and  $\lim_{s \rightarrow \pm\infty} a(s, t) = \pm\infty$ , for  $T_{\pm}$  the periods of  $\gamma_{\pm}$  respectively, and assuming standard complex structure  $j$  on  $\mathbb{R} \times S^1$  and almost complex structure  $J$  adjusted to a chosen concordance between  $\lambda_{\pm}$ . In the non-equivariant setting standard transversality arguments (as in [8], [3]) establish that for generic  $J$  the space  $\hat{\mathcal{M}}(\gamma_+, \gamma_-)$  is a smooth oriented manifold (assuming once again a system of coherent orientations). Slight modifications of these arguments (see below) apply in our equivariant setting implying the same statement for generic almost complex structures  $\mathbb{Z}_k$ -adjusted to a  $\mathbb{Z}_k$ -concordance.

The manifold  $\hat{\mathcal{M}}(\gamma_+, \gamma_-)$  is acted upon freely by the re-parametrization<sup>9</sup> group  $\mathbb{R} \times S^1$  and quotienting yields a smooth manifold  $\mathcal{M}(\gamma_+, \gamma_-)$  of dimension  $\mu_{CZ}(\gamma_+) - \mu_{CZ}(\gamma_-)$ , which compactifies to a moduli space of broken  $J$ -holomorphic cylinders by [4]. When  $\mu_{CZ}(\gamma_+) = \mu_{CZ}(\gamma_-)$  the space  $\mathcal{M}(\gamma_+, \gamma_-)$  is a finite collection of points with sign which we count modulo  $k$  and the chain map  $\text{mon} : C^{(a,b)}(\lambda_+, J) \rightarrow C^{(a,b)}(\lambda_-, J)$  is defined, exactly as in [11], by setting  $\text{mon}(\gamma_+)$  to be the sum over all  $\gamma_-$  weighted by this count  $\#\mathcal{M}(\gamma_+, \gamma_-)$ . This induces the (grading-preserving) monotonicity morphism  $\text{mon} : CH^{\mathbb{Z}_k, (a,b)}(\lambda_+, J) \rightarrow CH^{\mathbb{Z}_k, (a,b)}(\lambda_-, J)$ .

Three natural grading-preserving morphisms, besides  $\text{mon}$ , are important - those due to scaling invariance, contactomorphism invariance ( $\psi_{\sharp}$ ) and window enlargement. We’ve given  $\psi_{\sharp}$ . The other two are also immediate in both equivariant and non-equivariant settings from the definitions. Assume  $c > 1$ , then there

<sup>9</sup>(there is no longer invariance of  $J$  in the  $s$ -direction)

are chain maps

$$\begin{aligned} c_* : C^{\mathbb{Z}_k, (0, cb)}(c\lambda, J_c) &\xrightarrow{\cong} C^{\mathbb{Z}_k, (0, b)}(\lambda, J) \\ \text{win} : C^{\mathbb{Z}_k, (0, b)}(\lambda, J) &\rightarrow C^{\mathbb{Z}_k, (0, cb)}(\lambda, J) \end{aligned}$$

where the second map is an isomorphism in all gradings under the additional hypothesis that  $\lambda$  has no closed Reeb orbits with action in the window  $[b, cb]$ , and  $J_c$  denotes the re-scaled version of  $J$  adjusted to  $c\lambda$ . In general, the composition of  $c_*$  and  $\text{win}$ , in either order, gives  $\text{mon}$  (see Remark 4.40 in [11] which applies verbatim in our setting). In particular, this allows to extend the definition of  $\text{mon}$  to the case  $\lambda_+ \geq \lambda_-$  by first scaling  $\lambda_+$  by a suitably small factor  $c > 1$ . Moreover, by definition, concordances behave well under compactly supported contactomorphisms  $\psi$  since these preserve the contact forms at infinity and thus induce a bijective correspondence between concordances. Indeed, given contact forms  $\lambda_{\pm} = H_{\pm}(dt - \alpha_L)$  and induced forms  $\lambda'_{\pm} = \psi_*\lambda_{\pm} = (\psi^*)^{-1}\lambda_{\pm}$ , let  $F : W_{a_-}^{a_+} \rightarrow \mathbb{R}_+$  be a function defining a concordance structure between  $\lambda_{\pm}$ . Then the function  $G : W_{a_-}^{a_+} \rightarrow \mathbb{R}_+$ ,  $(v, s) \mapsto F(\psi^{-1}(v), s)/h(\psi^{-1}(v))$  defines a concordance structure between  $\lambda'_{\pm}$  where  $h : V \rightarrow \mathbb{R}_+$  such that  $\psi^*(dt - \alpha_L) = h(dt - \alpha_L)$  and  $\omega_G$  is induced from  $\omega_F$  by  $\psi \times \mathbb{1} : W_{a_-}^{a_+} \rightarrow W_{a_-}^{a_+}$ . This implies that  $\text{mon}$  commutes with  $\psi_{\#}$  on chain level. The identical argument applies in the equivariant setting assuming  $\psi \in \text{Cont}^{\mathbb{Z}_k}(\mathbb{R}^{2n} \times S^1)$  and  $\lambda_{\pm}$  is  $\mathbb{Z}_k$ -admissible. Passing to homology, the morphism  $\lambda_+ \preceq \lambda_-$  induces a morphism  $\text{mon} : CH_*(\lambda_+) \rightarrow CH_*(\lambda_-)$  which commutes with  $\psi_{\#}$ , and likewise in the equivariant case, for  $\psi$  in the respective group. In principle  $\text{mon}$  still depends on both the choice of concordance  $F$  and of adjusted almost complex structure  $J$ , while  $CH_*(\lambda, J)$ ,  $CH_*^{\mathbb{Z}_k}(\lambda, J)$  and  $\text{win}$  depend on  $J$ . By a straightforward argument (see Proposition 4.30 of [11]) which goes through verbatim in our setting the vector spaces  $CH_*(\lambda, J)$ ,  $CH_*^{\mathbb{Z}_k}(\lambda, J)$  are independent of  $J$ . Finally, by standard Floer theoretic arguments applied to a homotopy between concordances or between almost complex structures adjusted to a concordance (see page 1692 of [11]), it follows that monotonicity morphisms  $\text{mon}$  are independent of the choices of  $F$  and  $J$ . We omit  $J$  in the notation from now on.

**Invariants of domains.** Let  $U \in \mathcal{U}_k$ . The morphisms  $\text{mon}$  make the family of vector spaces  $\{CH^{\mathbb{Z}_k, (0, \epsilon)}(\lambda)\}_{\lambda \in \mathcal{F}_{ad}(U, \epsilon)}$  into a directed system over  $\mathcal{F}_{ad}(U, \epsilon)$ ,  $\preceq$  and after taking its direct limit, the morphisms  $\text{win}$  make resulting vector spaces  $\{CH^{\mathbb{Z}_k, (0, \epsilon)}(U)\}_{\epsilon \in \mathbb{R}_+}$  into an inverse system over  $\mathbb{R}_+$ ,  $\geq$ . Following [11] we define

$$CH^{\mathbb{Z}_k}(U) := \varprojlim_{\epsilon \rightarrow 0} \varinjlim_{\lambda \in \mathcal{F}_{ad}(U, \epsilon)} CH^{\mathbb{Z}_k, (0, \epsilon)}(\lambda)$$

and make the same definition without superscripts  $\mathbb{Z}_k$  for  $U \in \mathcal{U}$ . Then given  $U_1 \subset U_2$  there is an induced morphism  $\iota_* : CH^{\mathbb{Z}_k}(U_1) \rightarrow CH^{\mathbb{Z}_k}(U_2)$  and the properties of  $\text{mon}$  and  $\psi_{\#}$  pass to the double limit yielding the following statement in terms of  $\mathcal{G}$ -functors<sup>10</sup>, stated for the non-equivariant setting on p. 1650 of [11] (see also their Theorem 4.47):

<sup>10</sup>Repeating footnote 3 of [11]: Given a group  $\mathcal{G}$  acting on category  $\mathcal{U}$ , a  $\mathcal{G}$ -functor on  $\mathcal{U}$  is a functor  $F$  and a family of natural transformations  $g_{\#} : F \rightarrow F \circ g$ ,  $g \in \mathcal{G}$ , such that  $(gh)_{\#} = g_{\#} \circ h_{\#}$  for all  $g, h \in \mathcal{G}$ ; see also the reference to Jackowski and Slominska in [11]

**Theorem 3.3.**  $CH^{\mathbb{Z}_k}(-)$  is a  $\mathcal{G}$ -functor from  $\mathcal{U}_k$  to  $\mathbb{Z}$ -graded vector spaces, for  $\mathcal{G} = \text{Cont}^{\mathbb{Z}_k}(\mathbb{R}^{2n} \times S^1)$ .

**Technical arguments.** Whereas Eliashberg-Kim-Polterovich [11] restricted to  $\mathbb{Z}_2$ -coefficients in order to streamline their presentation avoiding the need for coherent orientations, these can be produced by the now-standard construction of Bourgeois-Mohnke [5] (see also Bourgeois-Oancea [6]) which pulls back orientations from the orientations of determinant line bundles over certain spaces of Fredholm operators. When all ingredients in the construction are  $\mathbb{Z}_k$ -invariant, the orientations on these bundles are as well, hence so too are their pull-backs. This gives coherent orientations in our equivariant setting and also means  $CH_*(-)$  of [11] is well-defined with  $\mathbb{Z}_k$ -coefficients; we re-prove their results for  $\hat{B}(R)$  using  $\mathbb{Z}_k$ -coefficients,  $k > 2$  in the next Section.

The compactification results used to define the differential in [11] are obtained as a consequence of the restriction on the action of closed contractible Reeb orbits for admissible contact forms and the requirement that almost complex structures be standard outside the symplectization of a compact subset  $\Xi$  of  $V$ . This second condition means  $S\Xi$  is foliated by weakly  $J$ -convex hypersurfaces so all  $J$ -holomorphic cylinders project to  $\Xi$ . Given this and the absence of contractible closed Reeb orbits in  $\mathcal{P}_\lambda$ , the results from [4] imply the needed compactification and glueing statements proving  $d^2 = 0$ . The above goes through verbatim in our equivariant setting as well.

Finally, we address transversality. Because the  $\mathbb{Z}_k$ -action we consider is a covering space action, only minor modifications of the standard arguments for the non-equivariant setting of [11] are required. By contrast, Milin's  $\mathbb{Z}_k$ -equivariant theory requires more intricate arguments because her  $\mathbb{Z}_k$ -action, generated by  $(z, t) \mapsto (e^{2\pi i/k} z, t)$ , is not free (it fixes all points of  $\{0\} \times S^1$ ).

The standard transversality arguments (see [3] for a detailed account) can be summarized as follows. To show the moduli space  $\hat{\mathcal{M}}(\gamma_+, \gamma_-) = \hat{\mathcal{M}}_J(\gamma_+, \gamma_-)$  which depends on a specific  $J$  is, for generic  $J$ , a Banach manifold of specified dimension, one considers, in the setting of [11], the universal moduli space  $\hat{\mathcal{M}}(\gamma_+, \gamma_-, \mathcal{I})$  where  $\mathcal{I}$  is the smooth Banach manifold of almost complex structures adjusted to  $\lambda$ , resp. adjusted to a concordance. This space,  $\hat{\mathcal{M}}(\gamma_+, \gamma_-, \mathcal{I})$ , can be identified with the zero set of the Cauchy-Riemann operator  $\bar{\partial}$  defined on the product  $\mathcal{B} \times \mathcal{I}$ , where  $\mathcal{B}$  is the smooth Banach manifold consisting of cylinder maps satisfying all conditions for elements of  $\hat{\mathcal{M}}(\gamma_+, \gamma_-)$  except the Cauchy-Riemann equation. By an infinite dimensional implicit function theorem, it suffices to show that the linearized operator  $L_{(G, J)} \bar{\partial}$  is surjective for all  $(G, J)$  in  $\hat{\mathcal{M}}(\gamma_+, \gamma_-, \mathcal{I})$ , then it follows that  $\hat{\mathcal{M}}(\gamma_+, \gamma_-, \mathcal{I})$  is a Banach manifold and so  $\hat{\mathcal{M}}_J(\gamma_+, \gamma_-)$  is as well for all regular values  $J$  of the projection map  $\pi : \hat{\mathcal{M}}(\gamma_+, \gamma_-, \mathcal{I}) \rightarrow \mathcal{I}$ , while, by Sard-Smale, the regular  $J$  are generic. In our  $\mathbb{Z}_k$ -equivariant setting we consider instead analogous spaces  $\mathcal{I}^k$  consisting of  $\mathbb{Z}_k$ -adjusted almost complex structures. The only part of the above argument we must modify is the proof that  $L_{(G, J)} \bar{\partial}$  is surjective for all  $(G, J)$  since in the  $\mathbb{Z}_k$ -equivariant setting this operator is now defined on the smaller space  $\mathcal{B} \times \mathcal{I}^k$  where  $\mathcal{I}^k$  consists of  $\mathbb{Z}_k$ -adjusted almost complex structures. In fact surjectivity here follows from that on  $\mathcal{B} \times \mathcal{I}$  by a covering space argument for  $\mathcal{I}^k$  as a  $k$ -fold

cover of  $\mathcal{I}$  but we instead describe how the surjectivity argument on  $\mathcal{B} \times \mathcal{I}$  can be directly modified.

Surjectivity in the specific case where  $\gamma_- = \gamma_+$  and  $G$  is a vertical cylinder is immediate, in equivariant as well as non-equivariant settings, since  $L_{(G,J)}\bar{\partial}$  decomposes as a direct sum of surjections (see [3]). For other  $G$ , in the non-equivariant setting surjectivity of  $L_{(G,J)}\bar{\partial}$  is obtained by showing that for any  $J \in \mathcal{I}$ , every non-trivial, finite energy  $J$ -holomorphic cylinder  $G = (f, a)$  in  $W$  resp.  $\bar{W}$  has an *injective point*  $p \in \mathbb{R} \times S^1$  [8], [3]. This is a point  $p$  such that  $df_p \neq 0$  and  $f^{-1}(f(p)) = \{p\}$ . For genericity of almost complex structures  $J$  adjusted to concordances a similar statement holds, and likewise for homotopies  $J_r$ ,  $r \in [0, 1]$  of almost complex structures adjusted to a single concordance (one can perturb to a homotopy of regular structures, fixing endpoints). In all three cases, the existence of needed injective points is guaranteed by the fact that near the ends of each holomorphic cylinder  $f$  restricts to an embedding [13]. One injective point implies a neighbourhood of such in every  $J$ -holomorphic cylinder  $G$  and this forces  $L_{(G,J)}\bar{\partial}$  to be surjective. In all three cases, to pass from existence of an injective point  $p$  on every cylinder  $G$  to surjectivity of  $L_{(G,J)}\bar{\partial}$  for all  $(G, J)$  the key ingredient is that one can choose a tangent vector  $X \in T_J\mathcal{I}$  supported in a small ball around  $F(p)$  with complete freedom due to injectivity of  $f$  at  $p$ . This allows to derive a contradiction should  $L_{(G,J)}$  fail to be surjective. In our  $\mathbb{Z}_k$ -equivariant framework where we work with  $\mathcal{I}^k$ , in order to carry this argument out it suffices to find a stronger kind of injective point, one which also satisfies  $f^{-1}(\{f(p), \nu f(p), \dots, \nu^{k-1}f(p)\}) = \{p\}$ . We call such a point  $\mathbb{Z}_k$ -*injective*<sup>11</sup>. Its existence means a tangent vector  $X \in T_J\mathcal{I}^k$  with support in  $\cup_{j=0}^k \nu^j(B)$  can still be chosen with complete freedom given a small ball  $B$  at  $p$  and so the usual argument (see [3]) goes through verbatim.

First we remark that when  $G$  is  $\mathbb{Z}_k$ -equivariant for the  $\mathbb{Z}_k$ -action on  $\mathbb{R} \times S^1$  generated by a  $1/k$ -shift in the  $S^1$  factor, then it is the lift to the  $k$ -fold cover of a  $J_0$ -holomorphic cylinder  $G_0 : \mathbb{R} \times S^1 \rightarrow V \times \mathbb{R}$  from the base of  $\mathbb{R} \times S^1$  to the non- $\mathbb{Z}_k$ -invariant base  $V \times R$  of  $\bar{W}$  resp.  $W$  where  $J_0$  is the non- $\mathbb{Z}_k$ -invariant almost complex structure in the base which lifts to  $J$ . We know  $L_{(G_0,J_0)}\bar{\partial}$  is surjective, so  $L_{(G,J)}\bar{\partial}$  is as well.

We then consider  $G$  which are not  $\mathbb{Z}_k$ -equivariant. In this case, we claim the image of  $G$  cannot be  $\mathbb{Z}_k$ -invariant. Indeed if it were then  $G$  and  $\nu G$  would be related by a holomorphic re-parametrization of  $(\mathbb{R} \times S^1, j)$ , but this is necessarily a translation and amounts asymptotically along  $\gamma_{\pm}$  to a shift by  $1/k$ , hence is exactly a shift by  $1/k$ , i.e.  $G$  is  $\mathbb{Z}_k$ -equivariant. Now, because  $G$  is not  $\mathbb{Z}_k$ -invariant we know  $\nu^j G(\mathbb{R} \times S^1) \neq G(\mathbb{R} \times S^1)$  for any  $j \in U(k)$ . At the same time, intersection points of holomorphic curves can accumulate only at critical values of both curves. We take a point  $p$  which is injective for  $G$  in the usual sense and consider a neighbourhood  $U$  of  $p$  consisting also of injective points (by openness of this condition). Since  $G(U)$  can intersect the other holomorphic cylinders  $\nu^j G(\mathbb{R} \times S^1)$ ,  $j \in U(k)$  at only finitely many points, there is necessarily a point  $p'$  (and hence small neighbourhood  $U' \ni p'$ ) which is  $\mathbb{Z}_k$ -injective.

<sup>11</sup>This definition is motivated by [16], but she also requires that  $f(p)$  avoid  $\{0\} \times S^1$ , a consideration not needed in our case

### 3.1 Results for $\widehat{B}(R)$ : application to non-squeezing

In this Section we state two results, Theorems 3.5 and 3.6, concerning  $\mathbb{Z}_k$ -equivariant contact homology of prequantized balls and show that together they imply our main result, Theorem 2.1. For comparison, we also state Theorem 3.4, a non-equivariant version of Theorem 3.5 which is a direct analog with  $\mathbb{Z}_k$ -coefficients of Eliashberg-Kim-Polterovich's result for  $CH_*(\widehat{B}(R))$  with  $\mathbb{Z}_2$ -coefficients (Theorem 1.28 and page 1721 of [11], with different grading convention). Proofs of these results are then given in Section 3.2. We remark that our proof<sup>12</sup> of Theorem 3.4 uses only contact homology, and does not pass to generalized Floer homology as did the proof of Eliashberg-Kim-Polterovich; it thus provides an alternative, more direct, proof of their result as well. We write  $[s]$  for the integer part of  $s \in (0, \infty)$ .

**Theorem 3.4** (c.f. Theorem 1.28 and page 1721 [11]). *When  $1/R \notin \mathbb{N}$*

$$CH_m(\widehat{B}(R)) = \begin{cases} \mathbb{Z}_k, & \text{if } m = -n - 2n[1/R] \\ 0, & \text{otherwise.} \end{cases}$$

*Moreover, if  $[1/R_1] = [1/R_2]$  then the morphism induced by inclusion  $\widehat{B}(R_2) \subset \widehat{B}(R_1)$  is an isomorphism in the grading  $m = -n - 2n[1/R_1] = -n - 2n[1/R_2]$ .*

By comparison, for  $\mathbb{Z}_k$ -equivariant contact homology we have:

**Theorem 3.5.** *When  $R > 1/k$  and  $1/R \notin \mathbb{N}$*

$$CH_m^{\mathbb{Z}_k}(\widehat{B}(R)) = \begin{cases} \mathbb{Z}_k, & \text{if } m \geq -n - 2n[1/R] \\ 0, & \text{if } m < -n - 2n[1/R]. \end{cases}$$

*Moreover, if  $R_1 \geq R_2 > 1/k$  then the morphism induced by inclusion  $\widehat{B}(R_2) \subset \widehat{B}(R_1)$  is an isomorphism in all gradings  $m \geq -n - 2n[1/R_1]$ .*

Broadly speaking, the basic computations for  $CH_*(\widehat{B}(R))$  and  $CH_*^{\mathbb{Z}_k}(\widehat{B}(R))$  are similar in that - in both cases - we find cofinal sequences of contact forms  $\lambda$  such that the chain complex  $(C(\lambda)_*^{(0,\epsilon)}, d)$  is quasi-isomorphic to  $0 \rightarrow \mathbb{Z}_k \rightarrow 0$  with non-trivial chain module in degree  $m_0 = -n - 2n[1/R]$ . A major difference in the two cases, however, is that non-equivariant homology  $H_m(0 \rightarrow \mathbb{Z}_k \rightarrow 0)$  is non-trivial iff  $m = m_0$ , while  $\mathbb{Z}_k$ -equivariant homology  $H_m^{\mathbb{Z}_k}(0 \rightarrow \mathbb{Z}_k \rightarrow 0)$  is non-trivial iff  $m \geq m_0$ .

Not only can we conclude from Theorem 3.5 that the morphism  $CH_*^{\mathbb{Z}_k}(\widehat{B}(R)) \rightarrow CH_*^{\mathbb{Z}_k}(\widehat{B}(R_\dagger))$  induced by an inclusion  $\widehat{B}(R) \subset \widehat{B}(R_\dagger)$  is an isomorphism in all degrees for which both  $CH_*^{\mathbb{Z}_k}(\widehat{B}(R))$  and  $CH_*^{\mathbb{Z}_k}(\widehat{B}(R_\dagger))$  are non-trivial, a stronger,  $\psi$ -perturbed, result also holds:

**Theorem 3.6.** *Given  $\psi \in \text{Cont}(\mathbb{R}^{2n} \times S^1)$  and  $R_\dagger, R > 1$  such that  $\psi(\widehat{B}(R)) \subset \widehat{B}(R_\dagger)$ , the inclusion morphism  $CH_*^{\mathbb{Z}_k}(\psi(\widehat{B}(R))) \rightarrow CH_*^{\mathbb{Z}_k}(\widehat{B}(R_\dagger))$  is an isomorphism in all degrees for which both  $CH_*^{\mathbb{Z}_k}(\widehat{B}(R))$  and  $CH_*^{\mathbb{Z}_k}(\widehat{B}(R_\dagger))$  are non-trivial.*

<sup>12</sup>Though we have formally restricted to  $k > 2$  to simplify the presentation of  $\mathbb{Z}_k$ -equivariant proofs (see Remark 3.1) this non-equivariant argument goes through without change for  $\mathbb{Z}_2$ -coefficients.

Given the  $\mathcal{G}$ -functoriality of  $CH_*^{\mathbb{Z}_k}(-)$  for  $\mathcal{G} = \text{Cont}^{\mathbb{Z}_k}(\mathbb{R}^{2n} \times S^1)$  (Theorem 3.3), Theorems 3.5 and 3.6 together imply Theorem 2.1:

of Theorem 2.1. Suppose  $\psi \in \text{Cont}_0^{\mathbb{Z}_k}(\mathbb{R}^{2n} \times S^1)$  squeezes  $\widehat{B}(R_1)$  into  $\widehat{B}(R_2)$  and  $R_2 < 1/\ell < R_1$ . By Remark 1.3 we may without loss of generality assume  $R_2$  to be as close to  $1/\ell$  as desired, in particular, assume  $1/k < R_2$ . By hypothesis  $[\frac{1}{R_1}] < \ell < [\frac{1}{R_2}]$ .

We have  $\psi(\widehat{B}(R_2)) \subset \psi(\widehat{B}(R_1)) \subset \widehat{B}(R_2)$ . By Theorem 3.6 the inclusion morphism  $CH_{p, \mathbb{Z}_k}(\psi(\widehat{B}(R_2))) \rightarrow CH_{p, \mathbb{Z}_k}(\widehat{B}(R_2))$  is an isomorphism for  $p$  in the grading range  $-n \geq p \geq -n - 2n[\frac{1}{R_2}]$ .

Let  $p = -n - 2n\ell$ . Then  $CH_{p, \mathbb{Z}_k}(\widehat{B}(R_1)) = 0$  by Theorem 3.5; hence,  $CH_{p, \mathbb{Z}_k}(\psi(\widehat{B}(R_1))) = 0$  by Theorem 3.3. This is a contradiction, as the isomorphism  $CH_{p, \mathbb{Z}_k}(\psi(\widehat{B}(R_2))) \rightarrow CH_{p, \mathbb{Z}_k}(\widehat{B}(R_2))$  must, by Theorem 3.3, factor through  $CH_{p, \mathbb{Z}_k}(\psi(\widehat{B}(R_1)))$ .  $\square$

**Remark 3.7** (Comparison with the argument of [11]). *The proof just given for Theorem 2.1 is in the same general spirit as Eliashberg-Kim-Polterovich's proof that  $\widehat{B}(R_1)$  cannot be squeezed into  $\widehat{B}(R_2)$  for  $R_2 < 1 < R_1$ . Their argument (generalized in Proposition 1.26 of [11]) uses the non-vanishing of an inclusion morphism  $CH_{-n}(\psi(\widehat{B}(R_1))) \rightarrow CH_{-n}(\widehat{B}(R_1))$  for large<sup>13</sup>  $R_1$  and  $CH_{-n}(\widehat{B}(R_2)) = 0$  both of which are given when  $R_2 < 1 < R_1$ . This strategy does not work verbatim in our setting because  $R_2 < R_1$  implies  $CH_m^{\mathbb{Z}_k}(R_1) \neq 0 \Rightarrow CH_m^{\mathbb{Z}_k}(\widehat{B}(R_2)) \neq 0$  in all gradings  $m \in \mathbb{Z}$ . A squeezing  $\psi$  of  $\widehat{B}(R_1)$  into  $\widehat{B}(R_2)$  implies, however, multiple interleavings,  $\psi(\widehat{B}(R_2)) \subset \psi(\widehat{B}(R_1)) \subset \widehat{B}(R_2) \subset \widehat{B}(R_1)$ , so one is not confined to have the “smaller  $R_j$ ” be the “monkey-in-the-middle”: we use the first pair of inclusions, while [11] use essentially the final pair. As we will see in Section 3.2, Theorem 3.6 which played a key role in our proof is also very much related to the isomorphism  $CH_{-n}^{\mathbb{Z}_k}(\psi(\widehat{B}(R_2))) \rightarrow CH_{-n}^{\mathbb{Z}_k}(\widehat{B}(R_1))$  which holds for large  $R_1$ .*

### 3.2 Computations for $\widehat{B}(R)$

of Theorems 3.4 and 3.5. To prove all parts of these Theorems (and later Theorem 3.6 as well) we use similar Hamiltonians. We first describe these in detail then give arguments for: (I) the first statements of both Theorems, (II) monotonicity morphisms when  $[1/R_1] = [1/R_2]$ , and (III) monotonicity morphisms when  $[1/R_1] < [1/R_2]$ .

Let  $R > 0$ , assume  $\epsilon > 0$  and fix  $\delta \in (0, 1)$ . Let  $m_0 := [1/R]$  (so  $1/(m_0+1)R < 1 < 1/m_0R$ ) and consider a 1-dimensional family<sup>14</sup> of piecewise-linear functions  $F := F_c$ ,  $c \in (1, \infty)$  as shown in Fig. 1 such that the value  $b \in (0, 1)$  is determined by  $c$  as specified below, and  $F$  consists of three linear pieces: a left-most linear piece of slope  $-\delta$  which passes through  $(0, c)$  and  $(1/\delta, 0)$ , a middle linear piece of twice that slope which passes through  $(b, 1)$ , and a right-most piece, on the

<sup>13</sup>They need  $R_1 > 1$  such that  $\widehat{B}(R_1)$  contains the support of  $\psi$ .

<sup>14</sup>These functions are similar to ones defined by [16] but we fix  $\delta$  and make other simplifications.

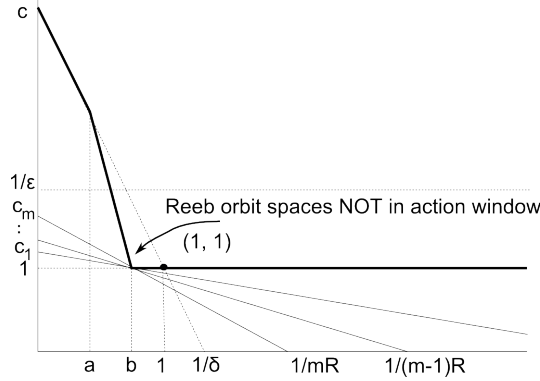


Figure 1: Graphs of Hamiltonians  $F(u) := F_c(u)$  on  $\widehat{B}(R)$  whose smoothings define contact forms  $(dt - \alpha_L)/F$  with a single closed Reeb orbit at  $\{0\} \times S^1$  of action  $1/c$  and no other closed Reeb orbits with action in the window  $(0, \epsilon)$ . Here  $m$  denotes  $m_0 = \lceil 1/R \rceil$  and we assume  $1 < 1/\delta < 1/m_0 R$  so there are no closed Reeb orbits corresponding to tangencies at  $(a, F(a))$ . Orbits for  $(b, F(b))$  on the other hand do not contribute when  $\epsilon$  is small since their action is bounded below by  $1/c_m$ .

interval  $(b, \infty)$  which is horizontal. The parameter  $b$  is required to be an increasing function  $b = f(c)$  of  $c$  such that  $\lim_{c \rightarrow \infty} f(c) = 1$ , and  $f(c+1) < f(c) + 1/2(c+1)\delta$  for sufficiently large  $c$ . For example  $f(c) = 1 - 1/c$  satisfies this inequality for  $c > 2\delta$ . A quick computation shows that the conditions on the middle segment of  $F_c$ , namely its slope and choice of  $b = f(c)$ , imply  $\frac{1}{2}F_{c+1} \leq F_c$ . Abusing notation, we define the Hamiltonian  $F_c$  on  $\mathbb{R}^{2n}$  to be the above function of the variable  $u = w/R$ , for  $w = \pi|z|^2$ . This lifts to an  $S^1$ -invariant Hamiltonian on  $\mathbb{R}^{2n} \times S^1$  equal to 1 outside  $\widehat{B}(R)$ ; when clear from context we will use the same name to refer to  $F_c$  as a function of the real variable  $u$  or as a Hamiltonian on  $\mathbb{R}^{2n}$  or  $\mathbb{R}^{2n} \times S^1$ . Put  $\mathbb{R}_C := (C, \infty)$  for suitable  $C > 2/\epsilon$  such that  $f(C) > 1/(m_0 + 1)R$ . Then  $f(c) > 1/(m_0 + 1)R$  for all  $c \in \mathbb{R}_C$  since the family  $F_c$  is non-decreasing in  $c$ . We consider the sequence  $\{F_i\}_{i \in \mathbb{N}_C}$  where  $\mathbb{N}_C := \mathbb{N} \cap \mathbb{R}_C$ . Among Hamiltonians which are constantly equal to 1 outside  $\widehat{B}(R)$  this is a dominating (i.e., cofinal) set.

We will now smooth each  $F_i$  in small neighbourhoods of its corners in such a way that the new tangent lines remain “sufficiently close” to those of the original  $F_i$  and the sequence of functions remains non-decreasing and dominating. To make these conditions precise, assume the smoothing operator  $A_\Delta$  has one parameter  $\Delta > 0$  which determines the fineness of the smoothing such that: (1) by decreasing  $\Delta$ ,  $A_\Delta(F)$  becomes arbitrarily  $C^0$ -close to  $F$ , (2) whenever  $F$  is linear on  $(u - \Delta, u + \Delta)$  then  $A_\Delta(F)(u) = F(u)$ , and (3) when  $(u - \Delta, u + \Delta)$  contains a single corner of  $F$ ,  $A_\Delta(F)(u) \geq F(u)$  with  $A_\Delta(F)(u)$  convex in the case  $F$  is convex over the interval, and  $A_\Delta(F)(u) \leq F(u)$  with  $A_\Delta(F)(u)$  concave in the case of concavity. Consider  $F := F_i$  for some  $i \in \mathbb{N}_C$ . For sufficiently small  $\Delta$ ,  $F^A := A_\Delta(F)$  is a positive, non-increasing function with

$F^A(u) = c(1 - \delta u)$  for small  $u$  and  $F^A(u) \equiv 1$  for  $u \geq 1$ . It is easily verified the contact form  $\lambda_{F^A} := (dt - \alpha_L)/F^A(u)$  has one closed Reeb orbit at  $\{0\} \times S^1$  of action  $1/F(0) = 1/c$  and spaces of closed Reeb orbits of action  $1/c_m$  along spheres of constant  $u$ -value such that  $-(F^A)'(u)/(F^A(u) - u(F^A)'(u)) = mR$  for some  $m \in \mathbb{N} \cup \{0\}$ . We denote each such  $u$ -value by  $u_m$  and the sphere  $\{u = u_m\}$  by  $S_m$ . In this case,  $c_m = -mR/(F^A)'(u_m)$ . On the other hand, for small  $\Delta$ , tangent lines to  $F^A$  are well approximated by so-called “generalized tangent lines” to the original piecewise linear  $F$ : these are either true tangent lines at points  $p$  of linearity or else lines of slope  $s \in [s_1, s_2]$  which pass through a corner point  $p$  where linear portions with slopes  $s_1 < s_2$  meet. As a result, one can read off the closed Reeb orbits of  $\lambda_{F^A}$  and their approximate actions from the graph of the un-smoothed  $F$  (c.f. Remark 5.2 [16]):  $F^A$  has one isolated orbit at  $\{0\} \times S^1$  with action  $1/c$  where  $c$  is the vertical intercept of  $F$ , and it has an  $S^{2n-1}$ -family of orbits with action arbitrarily close to  $1/c_m$  corresponding to each generalized tangent line to  $F$  at  $(b, F(b))$  that is horizontal ( $m = 0$ ) or has horizontal intercept  $1/mR > 0$  ( $m = 1, 2, \dots, m_0$ ) with  $c_m$  being in either case the vertical intercept of the generalized tangent line, always less than  $1/(1 - m_0R)$ .

If one imposes  $1/\delta < 1/m_0R$  then (with  $c > C$ ) this implies there are no generalized tangent lines at  $(a, F(a))$  with horizontal intercept  $1/mR$ ,  $R \in \mathbb{N}$  and, thus, no corresponding closed Reeb orbits (see Fig. 1). Otherwise, in general, there will be Reeb orbits corresponding to  $(a, F(a))$  and all have action greater than  $c$  (see Fig. 2). Let  $u = \bar{u}_m$  respectively be the  $u$ -values where such tangencies to the smoothing occur, and  $S_m := \{u = \bar{u}_m\} \subset \mathbb{R}^{2n}$  the corresponding spheres.

We now require the  $\Delta_i$  which defines a smoothing of  $F_i$ ,  $i \in \mathbb{N}_C$  to be sufficiently small that the true vertical intercepts of the tangent lines to the smoothing have the same ordering and lie on the same side of  $1/\epsilon$  as those of generalized tangent lines to  $F_i$ . Note we have fixed  $\epsilon > 0$  while computing  $\lim_{\lambda' \in \mathcal{F}_{ad}(\hat{B}(R), \epsilon)} CH^{(0, \epsilon)}(\lambda')$ . In addition  $\Delta_i$  should be sufficiently small that the  $\Delta_i$ -windows centred at corners of  $F_i$  do not overlap. Finally because  $\{F_i\}_{i \in \mathbb{N}_C}$  is non-decreasing we can take each  $\Delta_i$  small enough that the smoothing  $(F_i)^A := A_{\Delta_i}(F_i)$  will satisfy  $(F_{i-1})^A \leq (F_i)^A \leq F_{i+1}$  and because  $\frac{1}{2}F_i \leq F_{i-1} \leq F_i$  we can further reduce  $\Delta_i$  if necessary so that  $\frac{1}{2}(F_i)^A < (F_{i-1})^A \leq (F_i)^A$ . Inductively, the first condition implies the sequence  $\{(F_i)^A\}_{i \in \mathbb{N}_C}$  is non-decreasing; the second condition will be used to control monotonicity morphisms.

The above produces  $SU(n)$ -invariant Morse-Bott functions  $(F_i)^A$  on  $\mathbb{R}^{2n} = \mathbb{C}^n$  with critical submanifolds  $\{0\} \times S^1$  and the spheres  $S_m$  (if any). We now perturb each  $(F_i)^A$  following the Morse-Bott computational framework of Bourgeois [1] in order to obtain a Morse function  $\hat{F}_i$  on  $\mathbb{R}^{2n}$  which is  $\mathbb{Z}_k$ -invariant near each  $S_m$  (for a  $\mathbb{Z}_k$ -action to be specified) and for which we know certain parts of the associated contact homology chain complex  $CH^{(0, \epsilon)}(\lambda_{\hat{F}_i})$ . The perturbing functions are constructed as follows. Choose an arbitrarily  $C^0$ -small  $\mathbb{Z}_k$ -invariant Morse function  $g : S^1 \rightarrow \mathbb{R}$  with  $k$  maxima  $M_1, \dots, M_{k-1}$  and  $k$  minima  $m_1, \dots, m_{k-1}$ ,  $M_j = e^{2j\pi i/k}$  and  $m_j = e^{(2j+1)\pi i/k}$ , for the  $\mathbb{Z}_k$ -action generated by rotation through  $2\pi/k$ . Now consider an arbitrarily  $C^0$ -small Morse function  $f : \mathbb{CP}^{n-1} \rightarrow \mathbb{R}$  with one critical point of each even index  $2j$ ,  $j = 0, \dots, (n-1)$  and its pullback  $\pi^*f : S^{2n-1} \rightarrow \mathbb{R}$  via the Hopf bundle map  $\pi : S^{2n-1} \rightarrow \mathbb{CP}^{n-1}$ . This is a Morse-Bott function on  $S^{2n-1}$  with critical submanifolds which are iso-



lated Hopf circles. Perturb  $\pi^* f$  by adding to it a radially attenuated extension of  $g$  supported in a small tubular neighbourhood of each critical Hopf circle. This produces a Morse function  $h : S^{2n-1} \rightarrow \mathbb{R}$  with  $k$  critical points each of index  $2j$  and  $2j+1$  for  $j = 0, \dots, n-1$  and whose Morse complex is

$$0 \longrightarrow \mathcal{R} \xrightarrow{p(T)} \mathcal{R} \xrightarrow{(T-1)} \mathcal{R} \xrightarrow{p(T)} \dots \xrightarrow{(T-1)} \mathcal{R} \xrightarrow{p(T)} \dots \xrightarrow{(T-1)} \mathcal{R} \longrightarrow 0$$

where each arrow is multiplication by the specified polynomial  $(T-1)$  or  $p(T) = T^{k-1} + \dots + T + 1 \in \mathcal{R}$ . The function  $h$  is  $\mathbb{Z}_k$ -invariant for any<sup>15</sup>  $\mathbb{Z}_k$ -action on  $S^{2n-1} \subset \mathbb{C}^n$  generated by multiplication by  $e^{2\pi im/k}$ ,  $m \in U(k)$ . Fix a particular choice of  $m$  and re-label the  $2k$  critical points in every index if necessary so that permutation of generators of the chain modules  $\mathcal{R}$  induced by the chosen  $\mathbb{Z}_k$ -action on  $S^{2n-1}$  is given by multiplication by  $T$ . Then the above chain complex is the  $\mathbb{Z}_k$ -equivariant Morse complex for  $h$  for the chosen  $\mathbb{Z}_k$ -action on  $S^{2n-1}$ . Let  $\hat{F}_i$  denote the Morse function on  $\mathbb{R}^{2n}$  which is obtained by adding to  $(F_i)^A$  a radially attenuated extension of  $h$  supported in a small tubular neighbourhood  $\mathcal{N}_m$  of each critical sphere  $S_m$  of  $(F_i)^A$ . On each  $S_m \times S^1$  there are then  $2kn$  non-degenerate closed Reeb orbits for  $\lambda_{\hat{F}_i}$  which may be identified consistently with the  $2kn$  critical points of  $\hat{F}_i$  on  $S_m$ . The vertical shift contactomorphism  $\nu$  on  $\mathbb{R}^{2n} \times S^1$  moreover permutes these orbits in  $S_m \times S^1$  according to multiplication by  $e^{2\pi im/k}$  of the corresponding critical points of  $\hat{F}_i$  in  $S_m$ . Since  $[1/R] < k$ , the action is free, i.e.,  $m \in U(k)$  (this is important). As above, assume labeling of the  $k$  critical points of each index on  $S_m$  such that the associated permutation of these points as generators  $1, T, \dots, T^{k-1}$  of the chain module  $\mathcal{R}$  is given by multiplication by  $T$ . On the small neighbourhood  $\mathcal{N}_m$  of each  $S_m$ , the function  $\hat{F}_i$  is a  $\mathbb{Z}_k$ -equivariant Morse function for the  $\mathbb{Z}_k$ -action given by multiplication by  $e^{2\pi im/k}$ . By [1] the Conley-Zehnder indices of closed Reeb orbits for the contact form  $\lambda_{\hat{F}_i}$  are determined by Morse indices of corresponding critical points of  $\hat{F}_i$  and the differentials in the chain complex  $(C^{(0,\epsilon)}, d)(\lambda_{\hat{F}_i})$  corresponding to concordances between critical points in a common sphere  $S_m$  are the same as the corresponding differentials in the  $\mathbb{Z}_k$ -equivariant Morse complex of  $\hat{F}_i$ . When  $\epsilon > 0$  is sufficiently small, so that only closed Reeb orbits corresponding to  $(0, F_i(0))$  and  $(a, F_i(a))$  have action in the window  $(0, \epsilon)$ , then letting  $m_0 = [1/\delta]$  (which may be greater than  $[1/R]$ ) and  $\ell \in \mathbb{N} \cup \{0\}$  be such that  $m_0 + \ell = [1/R]$ , respective Conley-Zehnder indices of closed Reeb orbits for  $\hat{F}_i$  are  $-n - 2nm_0, 2n - 2n(m_0 + 1), \dots, 2n - 2n(m_0 + \ell)$  and the chain complex  $(C^{\mathbb{Z}_k, (0,\epsilon)}, d)(\lambda_{\hat{F}_i})$  is therefore  $0 \rightarrow \mathbb{Z}_k \xrightarrow{d_{m_0}} C[-n - 2nm_0] \xrightarrow{d_{m_0+1}} C[-n - 2n(m_0 + 1)] \rightarrow \dots \rightarrow C[-n - 2n(m_0 + \ell)] \rightarrow 0$  where  $C[j]$  denotes a sub-complex which is the Morse complex for  $h$  shifted in grading by  $j$ ; only the  $\mathbb{Z}_k$ -equivariant differentials  $d_{m+1}$  corresponding to concordances between Reeb orbits on adjacent spheres  $S_m$  and  $S_{m+1}$  are not immediately known. We assume once again (as for the operator  $A_\Delta$ ) that the perturbations used to construct  $\hat{F}_i$  are suitably  $C^0$ -small and carried out successively on  $(F_i)^A$ ,  $i \in \mathbb{N}_C$ .

(I) To simplify the notation put  $\lambda_i := \lambda_{(F_i)^A}$ . These contact forms, which

<sup>15</sup>The reader comparing with Milin's argument should note that she uses only  $m = 1$ ; we need vary  $m$  because our  $\mathbb{Z}_k$ -action induces on each  $S_m$  a different  $\mathbb{Z}_k$ -action, namely that generated by multiplication by  $e^{2\pi im/k}$ .

are  $S^1$ -invariant hence  $\mathbb{Z}_k$ -invariant, constitute a cofinal sequence not only for  $\mathcal{F}_{ad}^{\mathbb{Z}_k}(\widehat{B}(R), \epsilon)$  but also for  $\mathcal{F}_{ad}(\widehat{B}(R), \epsilon)$ . Moreover, each Morse-Bott function  $(F_i)^A$  has only one critical point, namely the origin, and so is already a  $\mathbb{Z}_k$ -invariant Morse function; thus,  $\widehat{F}_i = (F_i)^A$  and we retain the name  $(F_i)^A$ . Since  $\lambda_i$  has only the closed Reeb orbit at  $\{0\} \times S^1$  and this has Conley-Zehnder index  $-n$ , for all sufficiently small  $\epsilon > 0$  the chain complex  $(C^{(0, \epsilon)}, d)(\lambda_i)$  for each  $i \in \mathbb{N}_C$  is  $0 \rightarrow \mathbb{Z}_k \rightarrow 0$  with non-trivial chain module in degree  $-n$  and hence  $CH_*^{(0, \epsilon)}(\lambda_i)$  and  $CH_*^{\mathbb{Z}_k, (0, \epsilon)}(\lambda_i)$  satisfy the formulae we want for  $CH_*(\widehat{B}(R))$ , resp.  $CH_*^{\mathbb{Z}_k}(\widehat{B}(R))$ . It remains to check that monotonicity morphisms in both equivariant and non-equivariant contact homology are isomorphisms for chosen small  $\epsilon > 0$ ; the above result for specific  $\lambda_i$  will then pass to the double-limit. We give the argument for the non-equivariant case to fix notation but the equivariant case is identical. Fix  $i \in \mathbb{N}_C$ , and recall that  $\frac{1}{2}(F_{i+1})^A \leq (F_i)^A \leq (F_{i+1})^A$ . To avoid excessive subscripts put  $G = (F_{i+1})^A$ . On the one hand, there is a window enlargement isomorphism  $CH_*^{(0, \epsilon)}(\lambda_{\frac{1}{2}G}) \xrightarrow{\cong} CH_*^{(0, 2\epsilon)}(\lambda_{\frac{1}{2}G})$  because  $\frac{1}{2}G$  has no closed Reeb orbits with action in the window  $[\epsilon, 2\epsilon]$  (recall  $2/c < \epsilon$ ). On the other hand, using scaling invariance, we also have an isomorphism  $CH_*^{(0, 2\epsilon)}(\lambda_{\frac{1}{2}G}) = CH_*^{(0, 2\epsilon)}(2\lambda_G) \xrightarrow{\cong} CH_*^{(0, \epsilon)}(\lambda_G)$ . Composing these we obtain the monotonicity morphism  $\text{mon} : CH_*^{(0, \epsilon)}(\lambda_{\frac{1}{2}G}) \rightarrow CH_*^{(0, \epsilon)}(\lambda_G) = CH_*^{(0, \epsilon)}(\lambda_{i+1})$  is an isomorphism. Since it factors through  $\text{mon} : CH_*^{(0, \epsilon)}(\lambda_i) \rightarrow CH_*^{(0, \epsilon)}(\lambda_{i+1})$  and all vector spaces  $CH_m^{(0, \epsilon)}(\lambda_i)$  are either trivial or  $\mathbb{Z}_k$  this completes the proof of the first statements in Theorems 3.4 and 3.5.

(II) To prove the second statement of Theorem 3.4 we take essentially the same sequence of Hamiltonians for both  $\widehat{B}(R_1)$  and  $\widehat{B}(R_2)$ , however, as functions of their respective canonical coordinates  $u_1 := w/R_1$  and  $u_2 := w/R_2$  where  $w := \pi|z|^2$ . Remark that  $u_1 = 1/mR_1 \Leftrightarrow u_2 = 1/mR_2 \Leftrightarrow w = 1/m$  and the condition  $m_0 = [1/R_1] = [1/R_2]$  implies  $1/(m_0 + 1) < R_2 < R_1 < 1/m_0$  so  $1/(m_0 + 1)R_j < 1 < 1/m_0R_j$  for both  $j = 1, 2$ . Now let  $\delta_0 > m_0$  but still less than  $1/R_1 < 1/R_2$  and put  $\delta_1 = \delta_0R_1$ ,  $\delta_2 = \delta_0R_2$ . Then  $1/\delta_j \in (1, 1/m_0R_j)$  for both  $j = 1, 2$ . Define Hamiltonians  $F_c(u_1)$  for  $\widehat{B}(R_1)$  using  $\delta = \delta_1$ , and Hamiltonians  $H_c(u_2)$  for  $\widehat{B}(R_2)$  using  $\delta = \delta_2$ . It follows that  $H_c$  and  $F_c$  both coincide near  $\{0\} \times S^1$  with the Hamiltonian  $c(1 - \delta_0 u)$ . On the other hand, assume the same function  $f$  is chosen to define  $b = f(c)$  in both families,  $F_c$  and  $H_c$ . In the first case,  $f$  gives values of  $u_1$ ; in the second case of  $u_2$ . Since these are values at which lower corners of the respective Hamiltonian occur and  $u = u_jR_j$ ,  $j = 1, 2$  we see that  $F_c$ 's lower corner has larger  $u$ -value than  $H_c$ 's and so  $H_c \leq F_c$  for all  $c$ . As in (I) we now consider only  $c \in \mathbb{R}_C$  for suitable  $C > \epsilon/2$  such that  $f(c) > 1/(m_0 + 1)R_2$ . This implies  $f(c) > 1/(m_0 + 1)R_1$  and if necessary we adjust<sup>16</sup> the definition of  $f$  so that at least the minimal  $N \in \mathbb{N}_C$  satisfies  $f(N) < R_2/R_1$ . Now consider sequences  $F_i, H_i$ ,  $i \in \mathbb{N}_C$  and assume smoothings are done as above to yield  $SU(n)$ -invariant Morse-Bott functions  $(F_i)^A$  and  $(H_i)^A$  respectively. These each have a single critical point and so are already  $\mathbb{Z}_k$ -invariant Morse-functions. Put  $\lambda_i^F := \lambda_{(F_i)^A}$  and  $\lambda_i^H := \lambda_{(H_i)^A}$ . By the same argument as before, each  $\lambda_i^F$

<sup>16</sup>While  $f$  takes values between  $1/(m_0 + 1)R_2$  and 1 we can re-parametrize it so that an integer value  $N$  yields  $f(N)$  arbitrarily close to  $1/(m_0 + 1)R_1 < R_2/R_1 < 1$  as desired.

and each  $\lambda_i^H$  has the desired equivariant and non-equivariant contact homology and these pass to the double limit. We claim that for fixed  $\epsilon > 0$  sufficiently small and  $i \in \mathbb{N}_C$  sufficiently large the monotonicity morphism induced by  $\lambda_i^H \geq \lambda_i^F$  is an isomorphism, thus yielding the desired statement in the double limit. Indeed,  $(F_N)^A \leq (H_i)^A$  for sufficiently large  $i$  because  $\{(H_i)^A\}_{i \in \mathbb{N}_C}$  is a dominating sequence for  $\widehat{B}(R_2)$  and  $(F_N)^A(u_2) = 1$  for all  $u_2 \geq f(N)$  with  $f(N) < R_2/R_1$  (recall  $u_2 = R_2/R_1$  corresponds to  $u = R_2$ ). At the same time  $\lambda_N^F \geq \lambda_i^F$  induces an isomorphism in contact homology for all  $i \in \mathbb{N}_C$  by the arguments above so by functoriality and the fact that each  $CH_m(\widehat{B}(R))$  is either trivial or 1-dimensional, the inclusion morphism  $\iota_* : CH_m(\widehat{B}(R_2)) \rightarrow CH_m(\widehat{B}(R_1))$  is an isomorphism in all gradings  $m$ , in particular, the grading  $m = -n$  where both vector spaces are non-trivial. Note chain maps  $C^{\mathbb{Z}_k, (0, \epsilon)}(\lambda_i^H) \otimes E \rightarrow C^{\mathbb{Z}_k, (0, \epsilon)}(\lambda_i^F) \otimes E$  induced by the  $\mathbb{Z}_k$ -equivariant chain maps  $C^{\mathbb{Z}_k, (0, \epsilon)}(\lambda_i^H) \xrightarrow{\cong} C^{\mathbb{Z}_k, (0, \epsilon)}(\lambda_i^F)$  are then also isomorphisms in all gradings so, taking the double limit, we obtain moreover the special case  $[1/R_1] = [1/R_2]$  of the second statement of Theorem 3.5.

(III) Finally, to prove the second statement of Theorem 3.5 when  $[1/R_1] < [1/R_2]$ , consider once again Hamiltonians as above for  $\widehat{B}(R_1)$  using  $\delta = \delta_1$ , and for  $\widehat{B}(R_2)$  using  $\delta = \delta_2$ , where  $\delta_1/R_1 = \delta_0 = \delta_2/R_2$  for some choice of  $\delta_0$ . Let  $m_1 = [1/R_1]$ ,  $m_2 = [1/R_2]$ . Now, unlike before, we can no longer simultaneously

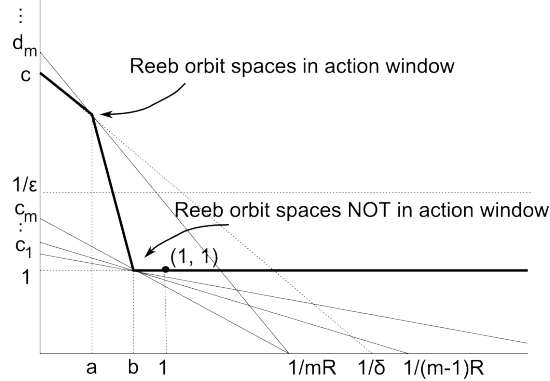


Figure 2: Graphs of Hamiltonians  $H(u) := H_c(u)$  as in Fig. 1 but where we allow  $1/\delta > 1/m_0 R$ , denoting  $m_0 = [1/R]$  once again by  $m$ . Smoothings of such  $H$  define contact forms  $\lambda = (dt - \alpha_L)/H$  which have not only the closed Reeb orbit  $\{0\} \times S^1$  with action in the window  $(0, \epsilon)$  as in Fig. 1, but additionally closed Reeb orbits corresponding to generalized tangent lines at  $(a, H(a))$ ; unlike orbits for tangencies at  $(b, H(b))$ , these orbits have action in the window  $(0, \epsilon)$ .

- for  $H_c$  and  $F_c$  - prevent generalized tangent lines at the upper corners which have horizontal intercept of the form  $1/mR$ . Indeed, taking  $\delta_0 > m_1$  (but less than  $1/R_1$ ) and  $C$  sufficiently large we prevent such lines for  $F_c$  as in Fig. 1, but then we necessarily have  $\delta_0 \not\geq m_2$  (because  $1/R_1 < 1/R_2 < m_2$ ) so such tangent lines will occur for  $H_c$  as in Fig. 2. We can guarantee, however, that their horizontal intercepts  $1/mR_2$  occur only for  $m$  such that  $m_1 < m < m_2$

by taking  $\delta_0 > m_1$  and restricting to  $c \in \mathbb{R}_C$  for some  $C > \epsilon/2$  such that  $f(c) > 1/(m_2 + 1)R_2$ . Note, for  $m_1 = m_2$  these conditions reduce to those imposed in paragraph (II). As before, take sufficiently large  $C$  and now consider perturbed smoothings  $\{\widehat{F}_i\}_{i \in \mathbb{N}_C}$  and  $\{\widehat{H}_i\}_{i \in \mathbb{N}_C}$  which are  $\mathbb{Z}_k$ -invariant Morse functions. Note: unlike in (II), the functions  $(H_i)^A$  were Morse-Bott with critical submanifolds  $S_m$ ,  $m_1 < m < m_2$ . Note  $m_2 < k$ . Now, fix  $\epsilon > 0$  and take  $i \in \mathbb{N}_C$  sufficiently large that  $F_i(0) = H_i(0) > 1/\epsilon$ . Let  $\eta > 0$  be a scaling factor such that the line  $y = \eta(1 - \delta u)$  is tangent to and lies below  $\widehat{H}_i$ . Then  $\eta\widehat{F}_i \leq \widehat{H}_i$  and since  $1/\delta_1 < 1/m_1 R_1$ , the action of all Reeb orbits corresponding to tangent lines to  $\eta\widehat{F}_i$  at points near the former “lower corner” of  $\widehat{H}_i$  will be strictly lower than  $\eta\widehat{F}_i(0)$ . Let  $\epsilon_0 < \eta\widehat{F}_i(0)$  be greater than these actions. Note that  $\text{mon} : C_*^{\mathbb{Z}_k, (0, \epsilon_0)}(\eta\widehat{F}_i) \rightarrow C_*^{\mathbb{Z}_k, (0, \epsilon_0)}(\widehat{F}_i)$  is then an isomorphism (by composing window enlargement and scaling invariance isomorphisms) and must factor through  $C_*^{\mathbb{Z}_k, (0, \epsilon_0)}(\widehat{H}_i)$  by functoriality. This implies the monotonicity morphism  $\text{mon} : C_j^{\mathbb{Z}_k, (0, \epsilon_0)}(\widehat{H}_i) \rightarrow C_j^{\mathbb{Z}_k, (0, \epsilon_0)}(\widehat{F}_i)$  is an isomorphism in grading  $j = -n - 2nm_1$  since all three chain modules are  $\mathbb{Z}_k$  in that grading. Note that shrinking  $\epsilon$  does not affect monotonicity morphisms except as generators of the respective chain modules appear or disappear. Since when we shrink  $\epsilon \rightarrow 0$  no closed Reeb orbits of Conley-Zehnder index  $-n - 2nm_1$  appear or disappear in chain modules  $C_j^{\mathbb{Z}_k, (0, \epsilon_0)}(\widehat{H}_i)$  and  $C_j^{\mathbb{Z}_k, (0, \epsilon_0)}(\widehat{F}_i)$ ,  $j = -n - 2nm_1$  we conclude that  $\text{mon} : C_j^{\mathbb{Z}_k, (0, \epsilon)}(\widehat{H}_i) \rightarrow C_j^{\mathbb{Z}_k, (0, \epsilon)}(\widehat{F}_i)$  is an isomorphism too in this grading. Because  $C_m^{\mathbb{Z}_k, (0, \epsilon)}(\widehat{F}_i) = 0$  in all other gradings  $m$ ,  $j$  is the only grading in which the chain map  $\text{mon}$  is non-trivial and by reasoning as in (II), one checks that the induced chain map  $C_*^{\mathbb{Z}_k, (0, \epsilon)}(\lambda_i^H) \otimes E \rightarrow C_*^{\mathbb{Z}_k, (0, \epsilon)}(\lambda_i^F) \otimes E$  consists of isomorphisms in row  $j$  of the double complex  $C_*^{\mathbb{Z}_k, (0, \epsilon)}(\lambda_i^H) \otimes E$  and vanishes in all other rows. This implies  $\text{mon} : CH_m^{\mathbb{Z}_k, (0, \epsilon)}(\widehat{H}_i) \rightarrow CH_m^{\mathbb{Z}_k, (0, \epsilon)}(\widehat{F}_i)$  is an isomorphism in all gradings  $m \geq -n - 2nm_1$ . Since this holds for all  $\epsilon > 0$  and  $i$  sufficiently large it passes to the double limit, implying the inclusion morphism  $\iota_* : CH_m^{\mathbb{Z}_k}(\widehat{B}(R_2)) \rightarrow CH_m^{\mathbb{Z}_k}(\widehat{B}(R_1))$  is an isomorphism for  $m \geq -n - 2nm_1$ .  $\square$

To prove Theorem 3.6 we use the following:

**Lemma 3.8.** *Assume a  $\mathbb{Z}_k$ -chain map*

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathbb{Z}_k & \xrightarrow{c_0^1 p} & \mathcal{R} & \xrightarrow{(T-1)} & \mathcal{R} & \xrightarrow{c_1^1 p} & \dots & \xrightarrow{(T-1)} & \mathcal{R} & \xrightarrow{c_N^1 p} & \dots & \xrightarrow{(T-1)} & \mathcal{R} & \longrightarrow & 0 \\
& & a_0 \downarrow & & a_1 \downarrow & & a_2 \downarrow & & & & & & a_{2N} \downarrow & & & & \\
0 & \longrightarrow & \mathbb{Z}_k & \xrightarrow{c_0^2 p} & \mathcal{R} & \xrightarrow{(T-1)} & \mathcal{R} & \xrightarrow{c_1^2 p} & \dots & \xrightarrow{(T-1)} & \mathcal{R} & \longrightarrow & 0
\end{array}$$

where each arrow is multiplication by the specified field element  $a_0 \in \mathbb{Z}_k$ , or polynomial  $(T-1) \in \mathcal{R}$ ,  $a_j = a_j(T) \in \mathcal{R}$  or  $c_i^j p \in \mathcal{R}$  for  $p = p(T) = T^{k-1} + \dots + T + 1 \in \mathcal{R}$  and  $c_i^j \in U(k)$ ,  $N = nm_1$ . Then, if  $a_0$  is a unit of  $\mathbb{Z}_k$ , all  $a_j$ ,  $j \in \{1, \dots, 2N\}$  are units of  $\mathcal{R}$ . Moreover, the conclusion also holds if the direction of the vertical arrows is reversed but all other hypotheses are the same.

This follows by induction<sup>17</sup> on  $j \in \mathbb{N} \cup \{0\}$  using the commutativity of  $\mathcal{R}$  and

<sup>17</sup>The author first encountered this observation in Milin [16].

the fact that  $(T - 1)$ , the maximal ideal of the local ring  $\mathcal{R}$ , contains all non-units and annihilates  $p(T)$ .

of Theorem 3.6. Fix  $\epsilon > 0$ . Let  $R_0 > 1$  be greater than both  $R$  and  $R_\dagger$  and also sufficiently large that  $\widehat{B}(R_0)$  contains the support of  $\psi$ . By Theorem 3.5 the monotonicity morphism  $\text{mon} : CH_{-n}(\widehat{B}(R)) \rightarrow CH_{-n}(\widehat{B}(R_0))$  is an isomorphism. Therefore by functoriality (Theorem 3.3)  $\text{mon} : CH_{-n}(\psi(\widehat{B}(R))) \rightarrow CH_{-n}(\widehat{B}(R_0))$  is an isomorphism and hence, using the fact these vector spaces are all equal to  $\mathbb{Z}_k$ ,  $\text{mon} : CH_{-n}(\psi(\widehat{B}(R))) \rightarrow CH_{-n}(\widehat{B}(R_\dagger))$  is an isomorphism too.

Assume dominating sequences of smoothed, perturbed Hamiltonians  $\{\widehat{H}_i\}_{i \in \mathbb{N}}$ ,  $\{\widehat{H}'_i\}_{i \in \mathbb{N}}$  and  $\{\widehat{F}_i\}_{i \in \mathbb{N}}$  for  $\widehat{B}(R)$ ,  $\widehat{B}(R_\dagger)$  and  $\widehat{B}(R_0)$  respectively as in part (III) of the previous proof where now the role of  $R_1$  is played by  $R_0$  for which  $0 = [1/R_0]$ . All contact forms  $\lambda_i^H$ ,  $\lambda_i^{H'}$  and  $\lambda_i^F$  therefore have a closed Reeb orbit at  $\{0\} \times S^1$  with Conley-Zehnder index  $-n$ ; the forms  $\lambda_i^F$  have only this orbit, while the forms  $\lambda_i^H$ ,  $\lambda_i^{H'}$  have others as well. More precisely, from the discussion of Morse-Bott computations in the previous proof (c.f. Bourgeois [1]) the chain complexes  $C_m^{\mathbb{Z}_k, (0, \epsilon)}(\lambda_i^H)$  and  $C_m^{\mathbb{Z}_k, (0, \epsilon)}(\lambda_i^{H'})$  are of the form  $0 \rightarrow \mathbb{Z}_k \xrightarrow{d_0} C[-n] \xrightarrow{d_1} C[-n-2n] \rightarrow \dots \xrightarrow{d_\ell} C[-n-2n\ell] \rightarrow 0$  for  $\ell$  respectively given by  $[1/R]$  or  $[1/R_\dagger]$ . Moreover, by  $\mathcal{G}$ -invariance,  $\mathcal{G} = \text{Cont}^{\mathbb{Z}_k}(\mathbb{R}^{2n} \times S^1)$ ,  $C_m^{\mathbb{Z}_k, (0, \epsilon)}(\psi_* \lambda_i^H)$  is of the same form. By comparison with the known non-equivariant contact homology of the respective domains one deduces that for sufficiently large  $i$ , all maps  $d_j$  are multiplication by  $c^j p(T) \in \mathcal{R}$  for  $p(T) = T^{k-1} + \dots + T + 1 \in \mathcal{R}$  and  $c^j \in U(k)$ .

By shifting the indexing of the sequence  $\{H'_i\}_{i \in \mathbb{N}_C}$  if necessary we may assume that  $H'_i \geq H_i \circ \psi$ , i.e.,  $\lambda_i^{H'} \leq \psi_* \lambda_i^H$ . There is thus a corresponding induced  $\mathbb{Z}_k$ -equivariant chain map  $\text{mon} : C_*^{\mathbb{Z}_k, (0, \epsilon)}(\psi_* \lambda_i^H) \rightarrow C_*^{\mathbb{Z}_k, (0, \epsilon)}(\lambda_i^{H'})$  which will be as shown in Lemma 3.8, but with all vertical arrows possibly directed upwards. Since  $\text{mon} : CH_{-n}(\psi(\widehat{B}(R))) \xrightarrow{\cong} CH_{-n}(\widehat{B}(R_\dagger))$  is an isomorphism and the vector spaces  $C_m^{\mathbb{Z}_k, (0, \epsilon)}(\lambda_i^{H'})$  and  $C_m^{\mathbb{Z}_k, (0, \epsilon)}(\psi_* \lambda_i^H)$  are  $\mathbb{Z}_k$  for all sufficiently large  $i$  the chain map  $\text{mon}$  in degree  $-n$  must be an isomorphism for all sufficiently large  $i$  and the result follows by applying Lemma 3.8 and passing to the double limit.  $\square$

## 4 Squeezing room and non-squeezing

By using/extending two of the constructions in Eliashberg-Kim-Polterovich [11] - namely, the map  $F_N : \mathbb{R}^{2n} \times S^1 \rightarrow \mathbb{R}^{2n} \times S^1$  defined for  $N \in \mathbb{Z}$  (see page 1649 of [11]),

$$(z, t) \mapsto (v(z)e^{2\pi N i t} z, t), \quad v(z) = \frac{1}{\sqrt{1 + N\pi|z|^2}}$$

and the squeezing given in Theorem 1.19 of [11] by a positive contractible loop of contactomorphisms of the ideal contact boundary  $P = S^{2n-1} \subset \mathbb{R}^{2n}$  - two observations follow.

First, that non-squeezing past  $R = m/\ell > 1$  is equivalent to a “squeezing room” requirement for squeezing past  $R = m/\kappa < 1$  which is in some cases

stronger than what is proved in [11]<sup>18</sup>. Second, even this stronger requirement may not be tight.

To see the first observation, note that  $F_b$  maps  $\widehat{B}(R)$  into  $\widehat{B}(R/(1+bR))$  for all  $R > 0$  and so in particular maps all of  $\mathbb{R}^{2n} \times S^1$  into  $\widehat{B}(1/b)$ , taking  $\widehat{B}(R)$  with  $R \in [1, \infty)$  to  $\widehat{B}(R')$  with  $R' \in [\frac{1}{b+1}, \frac{1}{b})$ . Thus, Theorem 1.2, resp. 1.2' is equivalent to

**Theorem 4.1.** *When*

$$\frac{1}{b+1} < \frac{m}{\kappa} < \frac{1}{b} \quad (1)$$

*there is no squeezing (resp. coarse squeezing) of  $\widehat{B}(m/\kappa)$  into itself within  $\widehat{B}(1/b)$ .*

In this sense, rigidity at large scale which completely precludes squeezing can be thought of as an infinite squeezing room requirement, and is equivalent to a form of rigidity at small-scale which requires squeezing room determined by the reciprocal integers. With this viewpoint, instead of a single cut-off between flexibility and rigidity of  $\widehat{B}(R)$  as  $R$  grows, one sees rather a squeezing room requirement which jumps at each reciprocal integer, culminating in an infinite requirement when  $R = 1$  is passed.

Theorem 1.5 of [11] established that there is no squeezing of  $\widehat{B}(m/\kappa)$  into itself within  $\widehat{B}(m/(\kappa - 1))$  but this bound depends on the particular  $m/\kappa$ . If equation (1) holds and  $m/\kappa$  is not too close to  $1/b$  - more precisely, if

$$\frac{1}{b+1} < \frac{m}{\kappa} < \frac{1}{b+1/m} \quad (2)$$

then we will have  $mb + 1 < \kappa$ , i.e.,

$$\frac{m}{\kappa - 1} < \frac{1}{b}$$

so the squeezing room requirement of Theorem 4.1 will (for such  $m/\kappa$ ) be strictly stronger than that of Theorem 1.5 of [11].

We now remark that even this stronger squeezing room requirement is not necessarily tight. If one applies Theorem 1.19 of [11] to  $\widehat{B}(m/\kappa)$ ,  $m < \kappa$  as done in Remark 1.23 of [11] to  $\widehat{B}(1/\kappa)$ , one obtains the more general:

**Proposition 4.2.** *There is a squeezing of  $\widehat{B}(m/\kappa)$  into itself within an arbitrarily small neighborhood of  $\widehat{B}(m/(\kappa - m))$ .*

However, (1) implies  $m/(\kappa - m) > 1/b$  so there remains a gap between the required squeezing room and that of known squeezings. To find squeezings with smaller support it seems methods beyond the construction of [11] would be needed, to produce either “wilder” compactly supported contact isotopies which do not deform the original fiberwise star-shaped domain through fiberwise star-shaped domains, or compactly supported contactomorphisms not isotopic to the identity.

---

<sup>18</sup>Statements in this Section for  $R = m/\kappa$  are to be compared with statements in [11] for  $R = m/k$ ; we have used the letter  $\kappa$  instead of  $k$  to avoid confusion with notation of preceding Sections. Considering the effect of the map  $F_b$ ,  $\kappa$  should be thought of as  $\ell + bm$  in this sentence.

**Future work** Besides closing the gap mentioned above another direction for future work is application of the framework of the present paper to prequantizations  $M \times S^1$  of other Liouville manifolds  $M$ . When  $M$  is a sufficiently stabilized Liouville manifold, squeezing at small scale has already been established by Eliashberg-Kim-Polterovich (c.f. Theorems 1.16, 1.19 in [11] of which Theorem 1.1 in our paper re-states the special case  $M = \mathbb{R}^{2n}$ ,  $n \geq 2$ ). We conjecture that coarse non-squeezing at large scale holds as well in these settings.

**Acknowledgements** I would like to express my deep gratitude to Leonid Polterovich for his mathematical guidance during several semesters of my studies (in Computer Science) at the University of Chicago, and for introducing me to the contact non-squeezing problem. None of the work in this paper would have started without the many discussions with him and this work became something of a second PhD for me, bringing me back into Mathematics. In particular the idea to tackle this problem by developing a  $\mathbb{Z}_k$ -equivariant form of contact homology is due to him. I owe a second debt of gratitude to Isidora Milin whose PhD thesis on orderability of lens spaces was an invaluable resource to me in developing the  $\mathbb{Z}_k$ -equivariant theory in this paper and also revealed enlightening similarities between the seemingly unrelated non-squeezing phenomena considered in her setting and ours. I also thank Frédéric Bourgeois for very helpful technical discussions. Finally I thank Yasha Eliashberg. His constant support and encouragement kept this project going despite countless delays. A portion of the writing of this work was completed while visiting Tel Aviv University in May 2014 and the Université de Lyon in June-July 2014. I thank both Leonid Polterovich and Jean-Yves Welschinger for their hospitality and for stimulating discussions. Those visits were partially supported by TAU and the CNRS respectively.

## References

- [1] F. Bourgeois, *A Morse-Bott approach to contact homology*. In “Symplectic and contact topology: interactions and perspectives”, eds. Y. Eliashberg, B. Khesin, F. Lalonde, Fields Inst. Commun. **35**, Amer. Math. Soc., 2003, pp. 55-77.
- [2] F. Bourgeois, *Introduction to Contact Homology*. Summer School in Berder: Holomorphic curves and contact topology, 2003.
- [3] F. Bourgeois, *Contact homology and homotopy groups of the space of contact structures*. Math. Res. Lett. **13**(1) (2006), 71-85.
- [4] F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki, and E. Zehnder, *Compactness results in symplectic field theory*, Geom. Topol. **7** (2003), 799-888.
- [5] F. Bourgeois, K. Mohnke, *Coherent orientations in symplectic field theory*. Math. Z. **248**(1) (2004), 123-146.
- [6] F. Bourgeois, A. Oancea, *Symplectic homology, autonomous Hamiltonians, and Morse-Bott moduli spaces*. arXiv:0704.1039 [math.SG] (2007).

- [7] S. Chiu, *Non-squeezing properties of contact balls*. [arXiv:1405.1178](#) [math.SG] (2014).
- [8] D. Dragnev, *Fredholm theory and transversality for noncompact pseudoholomorphic maps in symplectizations*. Comm. Pure Appl. Math. **57** (2004), 726-763.
- [9] Y. Eliashberg, A. Givental, and H. Hofer, *Introduction to symplectic field theory*. GAFA 2000. In “Visions in Mathematics”, eds. N. Alon, J. Bourgain, A. Connes, M. Gromov, V. Milman, Birkhäuser, 2000, pp. 560-673.
- [10] Y. Eliashberg, L. Polterovich, *Partially ordered groups and geometry of contact transformations*. GAFA **10** (2000), 1448-1476.
- [11] Y. Eliashberg, S. S. Kim, L. Polterovich, *Geometry of contact transformations and domains: orderability versus squeezing*. Geom. Topol. **10** (2006), 1635-1747.
- [12] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*. Invent. Math. **82** (1985), 307-347.
- [13] H. Hofer, K. Wysocki, E. Zehnder, *Properties of pseudoholomorphic curves in symplectizations. III Fredholm theory*. In “Topics in nonlinear analysis”, Progr. Nonlinear Differential Equations Appl. **35**, Birkhäuser, 1999, pp. 381-475.
- [14] J. Nelson, *Automatic transversality in contact homology I: regularity* [arXiv:1407.3993v2](#) [math.SG] (2015).
- [15] S. S. Kim, *Relative contact homology and contact nonsqueezing*, Ph.D. thesis, Stanford University, 2001.
- [16] I. Milin, *Orderability and Non-squeezing in Contact Geometry*. Ph.D. thesis, Stanford University, 2008.
- [17] J. Robbin and D. Salamon, *The Maslov index for paths*. Topol. **32**(4) (1993), 827-844.
- [18] D. Tamarkin, personal communication and talk, Geometry Seminar, University of Chicago, 2010.